Chapter 6. Application of Derivatives

RateMeasure Approximations and Increasing-Decreasing Functions

1 Marks Questions

1. The amount of pollution content added in air in a city due to x diesel vehicles is given by

$$P(x) = 0.005x^3 + 0.02x^2 + 30x.$$

Find the marginal increase in pollution content when 3 diesel vehicles are added and write which value is indicated in the above question? Delhi 2013; Value Based Question

Given, $P(x) = 0.005x^3 + 0.02x^2 + 30x$

On differentiating both sides w.r.t. x, we get

$$P'(x) = 3 \times 0.005x^2 + 2(0.02)x + 30$$

On putting $x = 3$, we get

$$P'(3) = 3 \times 0.005 \times 9 + 2(0.02)(3) + 30$$
$$= 0.135 + 0.12 + 30 = 30.255$$ **(1/2)**

Pollution content in the city increases with the increase in number of diesel vehicles. **(1/2)**

2. The total cost $C(x)$ associated with provision of free mid-day meals to x students of a school in primary classes is given by

$C(x) = 0.005x^3 - 0.02x^2 + 30x + 50.$

If the marginal cost is given by rate of change $\frac{dC}{dx}$ of total cost, then write the marginal cost of food for 300 students. What value is shown here?

Delhi 2013C; Value Based Question

Given, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$

On differentiating both sides w.r.t. x, we get

$$\frac{dC}{dx} = 0.005(3x^2) - 0.02(2x) + 30$$

$$= 0.015x^2 - 0.04x + 30$$

On putting $x = 300$, we get

$$\frac{dC}{dx} = 0.015(300)^2 - 0.04(300) + 30$$

$$= 1350 - 12 + 30 = 1368$$ **(1/2)**

By providing free mid-day meals to students of primary classes, care and concern is shown towards their health and nutritional status. **(1/2)**

3. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue).

If the total revenue (in ₹) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, then find the marginal revenue, when $x = 5$ and write which value does the question indicate.

All India 2012; Value Based Question

We know that, marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$\therefore$ Marginal revenue (MR) $= \frac{dR}{dx}$

$$= \frac{d}{dx}(3x^2 + 36x + 5) = 6x + 36$$

When $x = 5$, then

$$MR = 6(5) + 36 = 30 + 36 = 66$$

Hence, the required marginal revenue is ₹ 66.

(1/2)

More amount of money spent for the welfare of the employees with the increase in marginal revenue. **(1/2)**

4 Marks Questions

4. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is

(i) strictly increasing.

(ii) strictly decreasing. Delhi 2014

Given function is

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

On differentiating w.r.t. x, we get

$$f'(x) = 12x^3 - 12x^2 - 24x \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$12x^3 - 12x^2 - 24x = 0$$

$$\Rightarrow \quad 12x(x^2 - x - 2) = 0$$

$$\Rightarrow \quad 12\,x\,[x^2 - 2x + x - 2] = 0$$

$$\Rightarrow \quad 12\,x\,(x+1)(x-2) = 0$$

$$\Rightarrow \quad x = 0, -1 \text{ or } 2 \quad \textbf{(1)}$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = 12x(x+1)(x-2)$	Sign of $f'(x)$
$x < -1$	(–) (–) (–)	– ve
$-1 < x < 0$	(–) (+) (–)	+ ve
$0 < x < 2$	(+) (+) (–)	– ve
$x > 2$	(+) (+) (+)	+ ve

We know that, a function $f(x)$ is said to be strictly increasing, if $f'(x) > 0$ and it is said to be strictly decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is

(i) strictly increasing on the intervals $[-1, 0]$ and $[2, \infty)$.

(ii) strictly decreasing on the intervals $(-\infty, -1]$ and $[0, 2]$. **(2)**

5. Find the intervals in which the function given by $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36x}{5} + 11$ is

(i) strictly increasing.

(ii) strictly decreasing. All India 2014C

Given function is

$$f(x)=\frac{3x^4}{10}-\frac{4x^3}{5}-3x^2+\frac{36x}{5}+11$$

On differentiating both sides w.r.t. x, we get

$$f'(x)=\frac{12x^3}{10}-\frac{12x^2}{5}-6x+\frac{36}{5}+0 \quad \textbf{(1)}$$

On putting $f'(x)=0$, we get

$$\frac{12x^3}{10}-\frac{12x^2}{5}-6x+\frac{36}{5}=0$$

$$\Rightarrow \quad \frac{6}{5}\cdot(x-1)(x+2)(x-3)=0$$

$$\Rightarrow \quad x-1=0 \text{ or } x+2=0$$

$$\text{or} \quad x-3=0$$

$$\Rightarrow \quad x=-2, 1, 3 \quad \textbf{(1)}$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x)=\frac{6}{5}(x-1)(x+2)(x-3)$	**Sign of $f'(x)$**
$(-\infty, 2]$	$(-)(-)(-)$	$-$ ve
$[-2, 1]$	$(-)(+)(-)$	$+$ ve
$[1\ \ 3]$	$(+)(+)(-)$	$-$ ve
$[3, \infty)$	$(+)(+)(+)$	$+$ ve

We know that, a function $f(x)$ is said to be strictly increasing, if $f'(x)>0$ and decreasing, if $f'(x)<0$. So, the given function $f(x)$ is

(i) strictly increasing in $[-2,1]\cup[3,\infty)$.

(ii) strictly decreasing in $(-\infty,-2]\cup[1,3]$. **(2)**

6. Find the value(s) of x for which $y=[x(x-2)^2]$ is an increasing function.

All India 2014

Given function is $y = [x(x-2)]^2 = [x^2 - 2x]^2$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2(x^2 - 2x)\frac{d}{dx}(x^2 - 2x)$$

$$= 2(x^2 - 2x)(2x - 2) = 4x(x-2)(x-1) \quad \textbf{(1)}$$

On putting $\frac{dy}{dx} = 0$, we get

$4x(x-2)(x-1) = 0$

$\Rightarrow \quad x = 0, 1 \text{ and } 2$ **(1)**

Now, we find interval in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	Sign of $f(x)$	Nature of $y(x)$
$(-\infty, 0]$	$(-)(-)(-) = -$ ve	Strictly decreasing
$[0, 1]$	$(+)(-)(-) = +$ ve	Strictly increasing
$[1, 2]$	$(+)(-)(+) = -$ ve	Strictly decreasing
$[2, \infty)$	$(+)(+)(+) = +$ ve	Strictly increasing

Hence, y is increasing in $[0, 1]$ and $[2, \infty)$, i.e. $x \in (0,1)$ and $(2, \infty)$. **(2)**

7. Using differentials, find the approximate value of $(3.968)^{3/2}$. Delhi 2014C

Let $y = f(x) = (x)^{3/2}$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \frac{3}{2} \cdot x^{1/2} \quad \textbf{(1)}$$

Let $\quad x = 4 \quad$ and $\quad x + \Delta x = 3.968$

Then, $\quad \Delta x = -0.032$ **(1)**

Now, $f(x+\Delta x) = f(x) + f'(x)\Delta x$

$\therefore \quad (x+\Delta x)^{3/2} = (x)^{3/2} + \frac{3}{2}\cdot(x)^{1/2}\cdot(-0.032)$ **(1)**

$\Rightarrow (4-0.032)^{3/2} = (4)^{3/2} + \frac{3}{2}\cdot(4)^{1/2}\cdot(-0.032)$

$\Rightarrow \quad (3.968)^{3/2} = 8 + \frac{3}{2}\cdot 2\cdot(-0.032)$

$= 8 - 0.096 = 7.904$ **(1)**

8. Find the intervals in which the function

$$f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51 \text{ is}$$

(i) strictly increasing.

(ii) strictly decreasing. Foreign 2014C

Given function is

$$f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$$

On differentiating w.r.t. x, we get

$$f'(x) = 6x^3 - 12x^2 - 90x$$

$$= 6x(x^2 - 2x - 15) \qquad \textbf{(1)}$$

Now, on putting $f'(x) = 0$, we get

$$6x(x^2 - 2x - 15) = 0$$

$$\Rightarrow \quad 6x(x^2 - 5x + 3x - 15) = 0$$

$$\Rightarrow \quad 6x(x - 5)(x + 3) = 0$$

$$\Rightarrow \quad x = -3, 0, 5 \qquad \textbf{(1)}$$

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$\boldsymbol{f'(x) = 6x(x-5)(x+3)}$	**Sign of $\boldsymbol{f'(x)}$**
$(-\infty, -3]$	$(-)(-)(-)$	$-$ ve
$[-3, 0]$	$(-)(-)(+)$	$+$ ve
$[0, 5]$	$(+)(-)(+)$	$-$ ve
$[5, \infty)$	$(+)(+)(+)$	$+$ ve

(1)

We know that, a function $f(x)$ is said to be strictly increasing, if $f'(x) > 0$ and decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is

(i) strictly increasing in $[-3, 0] \cup [5, \infty)$.

(ii) strictly decreasing in $(-\infty, -3] \cup [0, 5)$. **(1)**

9. Find the approximate value of $f(3.02)$, upto 2 places of decimal, where $f(x) = 3x^2 + 5x + 3$.

Foreign 2014

Firstly, split 3.02 into two parts x and Δx, so that $x + \Delta x = 3.02$ and $f(x + \Delta x) = f(3.02)$.
Now, write $f(x + \Delta x) \approx f(x) + \Delta x \cdot f'(x)$ and use this result to find the required value.

Given function is $f(x) = 3x^2 + 5x + 3$

On differentiating w.r.t.x, we get $f'(x) = 6x + 5$

Let $x = 3$ and $\Delta x = 0.02$ **(1)**

So that $f(x + \Delta x) = f(3.02)$

By using $f(x + \Delta x) \approx f(x) + \Delta x f'(x)$, we get

$f(3.02) = 3x^2 + 15x + 3 + (6x + 15) \cdot \Delta x$ **(1)**

$= 3(3)^2 + 15(3) + 3 + [6(3) + 15](0.02)$ **(1)**

$= 27 + 45 + 3 + 33(0.02)$

$= 75 + 0.66 = 75.66$

Hence, $f(3.02) = 75.66$ **(1)**

10. Using differentials, find approximate value of $\sqrt{49.5}$. Delhi 2012

Firstly, divide 49.5 into two parts as $x = 49$ and $\Delta x = 0.5$. Now, let $y = \sqrt{x}$ and find $\frac{dy}{dx}$. Finally, find Δy using the formula $dy = \frac{dy}{dx} \cdot \Delta x$.

Let $x = 49$, $\Delta x = 0.5$ and $y = \sqrt{x}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \quad \textbf{(1)}$$

At $x = 49$, $\left[\frac{dy}{dx}\right]_{x=49} = \frac{1}{2\sqrt{49}} = \frac{1}{2 \times 7} = \frac{1}{14}$ **(1)**

We know that, $dy = \frac{dy}{dx} \cdot \Delta x$

$$\Rightarrow \quad dy = \frac{1}{14} \times 0.5 = \frac{5}{140} = \frac{1}{28} \quad \textbf{(1)}$$

$$\therefore \quad \sqrt{49.5} \approx y + \Delta y = \sqrt{49} + \frac{1}{28} = 7 + \frac{1}{28}$$

$$[\because y = \sqrt{x} = \sqrt{49} = 7]$$

$$= \frac{196 + 1}{28} = \frac{197}{28} = 7.035 \quad \textbf{(1)}$$

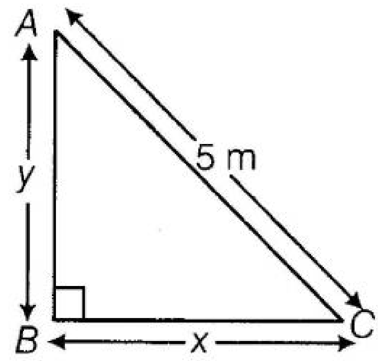
Hence, approximate value of $\sqrt{49.5}$ is 7.035.

11. A ladder 5 m long is leaning against a wall. Bottom of ladder is pulled along the ground away from wall at the rate of 2 m/s. How fast is the height on the wall decreasing, when the foot of ladder is 4 m away from the wall?

HOTS; All India 2012

The ladder when leaning against the wall forms a right angled triangle with length of ladder equal to the hypotenuse of the triangle and base and altitude equal to the distance of foot from wall and distance of top from ground. Further, use the Pythagoras theorem to make an equation and then differentiate it.

Let AC be the ladder, $BC = x$ and height of the wall, $AB = y$.



As the ladder is pulled along the ground away from the wall at the rate of 2 m/s. So,

$$\frac{dx}{dt} = 2 \text{ m/s}$$

To find $\frac{dy}{dt}$, when $x = 4$. **(1)**

In right-angled ΔABC, by Pythagoras theorem, we get

$$(AB)^2 + (BC)^2 = (AC)^2$$

$$\Rightarrow \quad x^2 + y^2 = 25 \quad \ldots\text{(i)}$$

$$\Rightarrow \quad (4)^2 + y^2 = 25$$

$$\Rightarrow \quad 16 + y^2 = 25 \Rightarrow y^2 = 9$$

$$\Rightarrow \quad y = \sqrt{9}$$

$$\therefore \quad y = 3$$ **(1)**

On differentiating both sides of Eq. (i) w.r.t. t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

[dividing both sides by 2]

On substituting the values of x, y and $\frac{dx}{dt}$, we get

$$(4 \times 2) + 3 \times \frac{dy}{dt} = 0$$

$$\Rightarrow \quad 8 + 3 \times \frac{dy}{dt} = 0$$ **(1)**

$$\frac{dy}{dt} = \frac{-8}{3} \text{ m/s}$$

Hence, height of the wall is decreasing at the rate of $\frac{8}{3}$ m/s. **(1)**

NOTE In a rate of change of a quantity, +ve sign shows that it is increasing and –ve sign shows that it is decreasing.

12. Show that $y = \log(1+x) - \dfrac{2x}{2+x}, x > -1$ is an increasing function of x, throughout its domain. **Foreign 2012**

Given function is $y = \log(1+x) - \dfrac{2x}{2+x}$.

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{1+x}(1) - \frac{(2+x)\cdot 2 - 2x\cdot 1}{(2+x)^2} \quad \textbf{(1)}$$

$$= \frac{1}{1+x} - \frac{4+2x-2x}{(2+x)^2}$$

$$= \frac{(2+x)^2 - 4(1+x)}{(1+x)(2+x)^2}$$

$$= \frac{4+x^2+4x-4-4x}{(1+x)(2+x)^2}$$

$$= \frac{x^2}{(1+x)(2+x)^2} \quad \ldots(i)$$

(1½)

Now, $x^2, (2+x)^2$ are always positive, also $1+x>0$ for $x>-1$. **(1/2)**

From Eq. (i), $\dfrac{dy}{dx} > 0$ for $x > -1$.

Hence, function increases for $x > -1$. **(1)**

13. Find the intervals in which the function given by $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$ is

(i) increasing. (ii) decreasing. **Delhi 2012C**

Given function is $f(x) = \sin x + \cos x$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x - \sin x \quad \textbf{(1)}$$

Now, put $f'(x) = 0$

$\Rightarrow \quad \cos x - \sin x = 0 \Rightarrow \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$, as $0 \le x \le 2\pi$ **(1)**

Thus, $f'(x) \ge 0$ in $\left[0, \frac{\pi}{4}\right]$, $f'(x) \le 0$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

and $f'(x) \ge 0$ in $\left[\frac{5\pi}{4}, 2\pi\right]$. **(1)**

Hence, the function is

(i) increasing in $\left[0, \frac{\pi}{4}\right]$ and $\left[\frac{5\pi}{4}, 2\pi\right]$.

(ii) decreasing in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. **(1)**

14. Find the intervals in which the function given by $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is

(i) increasing.

(ii) decreasing. All Delhi 2012C

Given function is

$$f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 4x^3 - 24x^2 + 44x - 24$$
$$= 4(x^3 - 6x^2 + 11x - 6)$$
$$= 4(x - 1)(x^2 - 5x + 6)$$
$$= 4(x - 1)(x - 2)(x - 3) \qquad \textbf{(1)}$$

Put $f'(x) = 0$

$$4(x - 1)(x - 2)(x - 3) = 0$$

$\Rightarrow$ $x = 1, 2, 3$

So, the possible intervals are $(-\infty, 1]$, $[1, 2]$, $[2, 3]$ and $[3, \infty)$. **(1)**

For interval $(-\infty, 1]$, $f'(x) \leq 0$

For interval $[1, 2]$, $f'(x) \geq 0$

For interval $[2, 3]$, $f'(x) \leq 0$

For interval $[3, \infty)$, $f'(x) \geq 0$

(i) Function increases in $[1, 2]$ and $[3, \infty)$. **(1)**

(ii) Function decreases in $(-\infty, 1]$ and $[2\ \ 3]$. **(1)**

15. Sand is pouring from the pipe at the rate of $12\ cm^3/s$. The falling sand forms a cone on a ground in such a way that the height of cone is always one-sixth of radius of the base. How fast is the height of sand cone increasing when the height is 4 cm? Delhi 2011

Let V be the volume of cone, h be the height and r be the radius of base of the cone.

Given, $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$ ...(i) **(1/2)**

Also, height of cone = 1/6 (radius of base of cone)

$$\therefore \quad h = \frac{1}{6} r \text{ or } r = 6h \quad \text{...(ii)} \ \textbf{(1/2)}$$

We know that, volume of cone is given by

$$V = \frac{1}{3} \pi r^2 h \quad \text{...(iii)}$$

On putting $r = 6h$ from Eq. (ii) in Eq. (iii), we get

$$V = \frac{1}{3} \pi (6h)^2 \cdot h \Rightarrow V = \frac{\pi}{3} \cdot 36h^3$$

$$\Rightarrow \quad V = 12\pi h^3 \quad \textbf{(1)}$$

On differentiating both sides w.r.t. t, we get

$$\frac{dV}{dt} = 12\pi \times 3h^2 \cdot \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = 36\pi h^2 \cdot \frac{dh}{dt} \quad \textbf{(1)}$$

On putting $\frac{dV}{dt} = 12 \text{ cm}^3/\text{s}$ and $h = 4$ cm, we get

$$12 = 36\pi \times 16 \times \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{12}{36\pi \times 16}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{48\pi} \text{ cm/s} \quad \textbf{(1)}$$

Hence, the height of sand cone is increasing at the rate of $1/48\pi$ cm/s.

16. Prove that $y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$ is an increasing function in $\left[0, \frac{\pi}{2}\right]$. **All India 2011**

To prove that given function is increasing, prove that $\frac{dy}{d\theta} \geq 0$ for all θ.

Given function is $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ ...(i)

We know that, a function $y = f(x)$ is said to be an increasing function, if $\frac{dy}{dx} \geq 0$, for all values of x.

On differentiating both sides of Eq. (i) w.r.t. θ, we get

$$\frac{dy}{d\theta} = \frac{\left[(2+\cos\theta)\times\frac{d}{d\theta}(4\sin\theta) - 4\sin\theta\times\frac{d}{d\theta}(2+\cos\theta)\right]}{(2+\cos\theta)^2} - 1 \quad \textbf{(1/2)}$$

$$= \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(0-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{\left[8\cos\theta + 4\cos^2\theta + 4\sin^2\theta - (2+\cos\theta)^2\right]}{(2+\cos\theta)^2}$$

$$= \frac{\left[8\cos\theta + 4(\cos^2\theta + \sin^2\theta) - (4+\cos^2\theta + 4\cos\theta)\right]}{(2+\cos\theta)^2} \quad \textbf{(1)}$$

$[\because (a+b)^2 = a^2 + b^2 + 2ab]$

$$= \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$[\because \sin^2\theta + \cos^2\theta = 1]$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \quad \textbf{(1)}$$

Now, as $\cos\theta \geq 0, \forall \theta \in \left[0, \frac{\pi}{2}\right]$.

and $(2+\cos\theta)^2$ being a perfect square is always positive for all $\theta \in \left[0, \frac{\pi}{2}\right]$.

Also, for $\theta \in \left[0, \frac{\pi}{2}\right]$, we know that,

$$0 \leq \cos\theta \leq 1.$$

$$\therefore \quad 4 - \cos\theta > 0 \text{ for all } \theta \in \left[0, \frac{\pi}{2}\right]$$

Hence, we conclude that

$$\frac{\cos\theta\,(4-\cos\theta)}{(2+\cos\theta)^2} \geq 0, \forall \theta \in \left[0, \frac{\pi}{2}\right].$$

$$\Rightarrow \quad \frac{dy}{d\theta} \geq 0, \forall \theta \in \left[0, \frac{\pi}{2}\right].$$

Hence, y is an increasing function in $\left[0, \frac{\pi}{2}\right]$. **(1½)**

17. If the radius of sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area. All India 2011

Let S be the surface area, r be the radius of the sphere.

Given $r = 9$ cm.

Let dr = Approximate error in radius r

and dS = Approximate error in surface area

(1)

Now, we know that surface area of sphere is given by

$$S = 4\pi r^2$$

On differentiating both sides w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi \times 2r = 8\pi r \qquad \textbf{(1)}$$

$$\Rightarrow \quad dS = 8\pi r \cdot dr$$

$$\Rightarrow \quad dS = 8\pi \times 9 \times 0.03$$

$$[\because r = 9 \text{ cm and } dr = 0 \cdot 03 \text{ cm}]$$

$$\Rightarrow \quad dS = 72 \times 0 \cdot 03\pi$$

$$\Rightarrow \quad dS = 2.16\pi$$

Hence, approximate error in surface area is 2.16π cm^2. **(2)**

18. Find the intervals in which the function $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

HOTS; Foreign 2011; All India 2009

For strictly decreasing function, $f'(x) < 0$ and for strictly increasing function, $f'(x) > 0$.

The given function is

$$f(x) = \sin x + \cos x, 0 \leq x \leq 2\pi$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x - \sin x \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$\cos x - \sin x = 0$$

$$\Rightarrow \quad \cos x = \sin x$$

$$\Rightarrow \quad 1 = \frac{\sin x}{\cos x}$$

$$\Rightarrow \quad 1 = \tan x$$

$$\Rightarrow \quad \tan x = \tan \frac{\pi}{4}$$

or $$\tan x = \tan \frac{5\pi}{4}$$

$$\left[\because \tan \frac{\pi}{4} = 1 \text{ and } \tan \frac{5\pi}{4} = 1\right]$$

$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \quad \textbf{(1)}$$

Now, we find intervals and check in which interval $f(x)$ is strictly increasing or strictly decreasing.

Interval	Test value	$f'(x) = \cos x - \sin x$	Sign of $f'(x)$
$0 < x < \frac{\pi}{4}$	At $x = \frac{\pi}{6}$	$\cos \frac{\pi}{6} - \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$	+ve
$\frac{\pi}{4} < x < \frac{5\pi}{4}$	At $x = \frac{\pi}{2}$	$\cos \frac{\pi}{2} - \sin \frac{\pi}{2} = 0 - 1 = -1$	–ve
$\frac{5\pi}{4} < x < 2\pi$	At $x = \frac{3\pi}{2}$	$\cos \frac{3\pi}{2} - \sin \frac{3\pi}{2} = 0 - (-1) = 1$	+ve

Since, $f'(x) > 0$ for $0 < x < \frac{\pi}{4}$ and $\frac{5\pi}{4} < x < 2\pi$, so $f(x)$ is strictly increasing in the intervals $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$. While $f'(x) < 0$ in $\frac{\pi}{4} < x < \frac{5\pi}{4}$, so $f(x)$ is strictly decreasing in the interval $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$. **(2)**

NOTE When we check that function f' is increasing/decreasing in an interval, we take any value in that interval and put them in function f' and check whether it is positive/negative.

19. Show that the function $f(x) = x^3 - 3x^2 + 3x$, $x \in R$ is increasing on R. **All India 2011C**

We know that, a function $y = f(x)$ is said to be increasing on R, if $\frac{dy}{dx} \geq 0, \forall x \in R$. **(1)**

Given, $y = x^3 - 3x^2 + 3x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 - 6x + 3 \Rightarrow \frac{dy}{dx} = 3(x^2 - 2x + 1)$$

$$\Rightarrow \frac{dy}{dx} = 3(x-1)^2 \quad \textbf{(1)}$$

Now, $3(x-1)^2 \geq 0$ for all real values of x, i.e. $\forall x \in R$.

$$\therefore \quad \frac{dy}{dx} \geq 0, \forall x \in R$$

Hence, the given function is increasing on R. **(2)**

20. Find the intervals in which the function $f(x) = (x-1)^3 (x-2)^2$ is

(i) increasing. (ii) decreasing. **All India 2011C**

Given, $f(x) = (x-1)^3 (x-2)^2$

On differentiating both sides w.r.t. x, we get

$$f'(x) = (x-1)^3 \cdot \frac{d}{dx}(x-2)^2 + (x-2)^2 \cdot \frac{d}{dx}(x-1)^3$$

$$\left[\because \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}\right]$$

$\Rightarrow f'(x) = (x-1)^3 \cdot 2(x-2) + (x-2)^2 \cdot 3(x-1)^2$

$= (x-1)^2 (x-2)[2(x-1) + 3(x-2)]$

$= (x-1)^2 (x-2)(2x - 2 + 3x - 6)$

$\Rightarrow f'(x) = (x-1)^2 (x-2)(5x-8)$

Now, put $f'(x) = 0$

$\Rightarrow \quad (x-1)^2 (x-2)(5x-8) = 0$

Either $(x-1)^2 = 0$ or $x - 2 = 0$ or $5x - 8 = 0$

$\therefore \quad x = 1, \frac{8}{5}, 2$ **(1)**

Now, we find intervals and check in which interval $f(x)$ is increasing and decreasing.

Interval	$f'(x) = (x-1)^2 (x-2)(5x-8)$	Sign of $f'(x)$
$x<1$	(+)(–)(–)	+ve
$1 < x < \frac{8}{5}$	(+)(–)(–)	+ve
$\frac{8}{5} < x < 2$	(+)(–)(+)	–ve
$x>2$	(+)(+)(+)	+ve

(1)

We know that, a function $f(x)$ is said to be an increasing function, if $f'(x) \geq 0$ and decreasing, if $f'(x) \leq 0$. So, the given function $f(x)$ is increasing on the intervals $(-\infty, 1]\left[1, \frac{8}{5}\right]$ and $(2, \infty]$ and decreasing on $\left[\frac{8}{5}, 2\right]$. **(1)**

Since, $f(x)$ is a polynomial function, so it is continuous at $x = 1, \frac{8}{5}, 2$. Hence, $f(x)$ is

(ii) increasing on intervals $(-\infty, 1], \left[1, \frac{8}{5}\right][2, \infty)$.

(ii) and decreasing on interval $\left[\frac{8}{5}, 2\right]$. **(1)**

21. Find the intervals in which the function $f(x) = 2x^3 + 9x^2 + 12x + 20$ is

(i) increasing. (ii) decreasing. **Delhi 2011C**

The given function is

$$f(x) = 2x^3 + 9x^2 + 12x + 20$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 6x^2 + 18x + 12$$

Put $f'(x) = 0$, we get

$$6x^2 + 18x + 12 = 0$$

$\Rightarrow$ $6(x^2 + 3x + 2) = 0$

$\Rightarrow$ $6(x + 1)(x + 2) = 0$

$\Rightarrow$ $(x + 1)(x + 2) = 0$

$\Rightarrow$ $x + 1 = 0$

or $x + 2 = 0$

$\therefore$ $x = -2, -1$ **(1)**

Now, we find intervals and check in which interval $f(x)$ is increasing and decreasing.

Interval	$f'(x) = 6(x+1)(x+2)$	Sign of $f'(x)$
$x < -2$	$(+)(-)(-)$	+ve
$-2 < x < -1$	$(+)(-)(+)$	−ve
$x > -1$	$(+)(+)(+)$	+ve

(1)

We know that, a function $f(x)$ is said to be an increasing function, if $f'(x) \geq 0$ and decreasing, if $f'(x) \leq 0$. So, given function is increasing on intervals $(-\infty, -2]$ and $[-1, \infty)$ and decreasing on interval $[-2, -1]$. **(1)**

Since, $f(x)$ is a polynomial function, so it is continuous at $x = -1, -2$.

Hence, given function is

(i) increasing on intervals $(-\infty, -2]$ and $[-1, \infty)$.

(ii) decreasing on interval $[-2, -1]$. **(1)**

22. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x - 15$ is
(i) increasing. (ii) decreasing. **Delhi 2011C**

The given function is

$$f(x) = 2x^3 - 9x^2 + 12x - 15$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 6x^2 - 18x + 12$$

On putting $f'(x) = 0$, we get

$$6x^2 - 18x + 12 = 0$$

$\Rightarrow$ $6(x^2 - 3x + 2) = 0$

$\Rightarrow$ $6(x-1)(x-2) = 0$

$\Rightarrow$ $x - 1 = 0$ or $x - 2 = 0$

$\therefore$ $x = 1, 2$ **(1)**

Now, we find intervals and check in which intervals $f(x)$ is increasing or decreasing.

Interval	$f'(x) = 6(x-1)(x-2)$	Sign of $f'(x)$
$x<1$	$(+)(-)(-)$	+ve
$1<x<2$	$(+)(+)(-)$	−ve
$x>2$	$(+)(+)(+)$	+ve

(1)

Now, we know that, a function $f(x)$ is increasing when $f'(x) \geq 0$ and it is decreasing when $f'(x) \leq 0$. So, the given function is increasing on intervals $(-\infty, 1)$ and $(2, \infty)$ and decreasing on intervals $(1, 2)$. **(1)**

Since, $f(x)$ is a polynomial function, so it is continuous at $x = 1, 2$.

Hence, given function is

(i) increasing on intervals $(-\infty, 1]$ and $[2, \infty)$.

(ii) decreasing on interval $[1, 2]$. **(1)**

23. Find the intervals in which the function $f(x) = 2x^3 - 15x^2 + 36x + 17$ is increasing or decreasing. **All India 2010C**

Do same as Que. 22.

[**Ans.** Increasing on $(-\infty, 2]$ and $[3, \infty)$ and decreasing on $[2,3]$.]

24. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is

(i) increasing. (ii) decreasing.

All India 2010C, 2008, 2008C

Do same as Que. 22.

[**Ans.** (i) Increasing on $(-\infty, 1]$ and $[2, \infty)$

(ii) Decreasing on $[1,2]$.]

25. Find the intervals in which the function $f(x) = (x-1)(x-2)^2$ is increasing or decreasing.

All India 2009C

The given function is $f(x) = (x-1)(x-2)^2$

On differentiating both sides w.r.t. x, we get

$$f'(x) = (x-1) \times \frac{d}{dx}(x-2)^2 + (x-2)^2 \times \frac{d}{dx}(x-1) \quad \textbf{(1)}$$

$$\Rightarrow \quad f'(x) = (x-1)\,2\,(x-2) + (x-2)^2 \cdot 1$$

$$\Rightarrow \quad f'(x) = 2(x-1)(x-2) + (x-2)^2$$

$$= (x-2)[2x - 2 + x - 2]$$

$$\Rightarrow \quad f'(x) = (x-2)(3x-4)$$

On putting $f'(x) = 0$, we get

$$(x-2)(3x-4) = 0$$

$$\Rightarrow \quad x - 2 = 0$$

$$\text{or} \quad 3x - 4 = 0$$

$$\therefore \quad x = \frac{4}{3} \text{ or } 2 \quad \textbf{(1)}$$

Now, we find the intervals in which $f(x)$ is increasing or decreasing.

Interval	$f'(x) = (x-2)(3x-4)$	Sign of $f'(x)$
$x < \frac{4}{3}$	(–) (–)	+ve
$\frac{4}{3} < x < 2$	(–) (+)	–ve
$x > 2$	(+) (+)	+ve

(1)

We know that, a function $f(x)$ is said to be an increasing function, if $f'(x) \geq 0$ and a decreasing function when $f'(x) \leq 0$. So, $f(x)$ is increasing on $\left(-\infty, \frac{4}{3}\right)$ and $(2, \infty)$ and decreasing on $\left(\frac{4}{3}, 2\right)$.

Since, $f(x)$ is a polynomial function, so it is continuous at $x = 4/3$ and 2.

Hence, given function is increasing on intervals $\left(-\infty, \frac{4}{3}\right]$ and $[2, \infty)$ and decreasing on interval $\left[\frac{4}{3}, 2\right]$. **(1)**

26. Find the intervals in which $f(x) = x^3 - 12x^2 + 36x + 17$ is increasing or decreasing function. **Delhi 2009C**

Do same as Que. 23.

[**Ans.** Increasing on $(-\infty, 2]$ and $[6, \infty)$ and decreasing on $[2, 6]$.]

27. The length x of a rectangle is decreasing at the rate of 5 cm/min and the width y is increasing at the rate of 4 cm/min. When $x = 8$ cm and $y = 6$ cm, find the rate of change of

(i) the perimeter. (ii) area of rectangle.

HOTS; All India 2009C

Using the relation perimeter of rectangle, $P = 2(x + y)$ and area of rectangle, $A = xy$, differentiate both sides with respect to t and put them in rate of change value and get the result.

Given that length x of a rectangle is decreasing at the rate of 5 cm/min.

$$\therefore \quad \frac{dx}{dt} = -5 \text{ cm/min} \quad \ldots(i)$$

Also, the breadth y of rectangle is increasing at the rate of 4 cm/min.

$$\therefore \quad \frac{dy}{dt} = 4 \text{ cm/min} \quad \ldots(ii)$$ **(1/2)**

(i) Here, we have to find rate of change of perimeter, i.e. dP/dt **(1)**

and we know that, perimeter $P = 2(x + y)$

On differentiating both sides w.r.t. t, we get

$$\frac{dP}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

28. Find the intervals in which the function f given by $f(x) = x^3 + \frac{1}{x^3}$, $x \neq 0$ is

(i) increasing.

(ii) decreasing. Delhi 2009C

Given function is $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 3x^2 - \frac{3}{x^4} \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$3x^2 - \frac{3}{x^4} = 0 \Rightarrow \frac{3x^6 - 3}{x^4} = 0$$

$$\Rightarrow 3x^6 - 3 = 0 \Rightarrow 3x^6 = 3$$

$$\Rightarrow x^6 = 1 \Rightarrow x = \pm 1$$

$$\left[\begin{array}{lll} \because x^6 = (1)^6 & \text{and} & x^6 = (-1)^6 \\ \Rightarrow x = 1 & \text{and} & x = -1 \end{array}\right] \textbf{(1)}$$

Now, we find intervals in which $f(x)$ is increasing or decreasing.

Interval	$f'(x) = \frac{3x^6 - 3}{x^4}$	Sign of $f'(x)$
$x < -1$	$\frac{(+)}{(+)}$	+ve
$-1 < x < 1$, $x \neq 0$	$\frac{(-)}{(+)}$	– ve
$x > 1$	$\frac{(+)}{(+)}$	+ve

Now, we know that a function $f(x)$ is increasing when $f'(x) \geq 0$ and it is said to be decreasing when $f'(x) \leq 0$. So, $f(x)$ is increasing on intervals $(-\infty, -1)$ and $(1, \infty)$ and it is decreasing on $(-1, 1) - \{0\}$.

Also, $f(x)$ is continuous at $x = 1, -1$.

Hence, $f(x)$ is

(i) increasing on intervals $(-\infty, -1]$ and $[1, \infty)$.

(ii) decreasing on interval $[-1, 1] - \{0\}$. **(1)**

29. If $f(x) = 3x^2 + 15x + 5$, then find the approximate value of $f(3.02)$ using differentials. Delhi 2008C

Do same as Que. 9. **[Ans.** 77.66]

6 Marks Questions

30. Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence, find the intervals in which $f(x)$ is

(i) strictly increasing. (ii) strictly decreasing.

Delhi 2014C

Given function is $f(x) = x^2 - x + 1$ **(1)**

On differentiating w.r.t. x, we get

$$f'(x) = 2x - 1$$ **(1)**

For $f(x)$ to be increasing, we must have

$$f'(x) > 0$$ **(1)**

$\Rightarrow$ $2x - 1 > 0 \Rightarrow x < 1/2$

So, $f(x)$ is decreasing on $(-\infty, 1/2]$. **(1)**

Hence, $f(x)$ is neither increasing nor decreasing in $[-1, 1]$. **(2)**

31. Find the intervals in which the function $f(x) = 20 - 9x + 6x^2 - x^3$ is

(i) strictly increasing. (ii) strictly decreasing.

All India 2010

The given function is $f(x) = 20 - 9x + 6x^2 - x^3$.

On differentiating both sides w.r.t. x, we get

$$f'(x) = -9 + 12x - 3x^2 \quad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$-9 + 12x - 3x^2 = 0$$

$\Rightarrow$ $-3(x^2 - 4x + 3) = 0$

$\Rightarrow$ $-3(x-1)(x-3) = 0$

$\Rightarrow$ $(x-1)(x-3) = 0$

$\Rightarrow$ $x - 1 = 0$ or $x - 3 = 0$

$\therefore$ $x = 1$ or 3 **(2)**

Now, we find intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	$f'(x) = -3(x-1)(x-3)$	Sign of $f'(x)$
$x < 1$	$(-)(-)(-)$	$-$ve
$1 < x < 3$	$(-)(+)(-)$	$+$ve
$x > 3$	$(-)(+)(+)$	$-$ve

(1)

Now, we know that, a function $f(x)$ is said to be strictly increasing when $f'(x) > 0$ and it is said to be strictly decreasing, if $f'(x) < 0$. So, the given function $f(x)$ is

(i) strictly increasing on the interval $(1, 3)$ and

(ii) strictly decreasing on the intervals $(-\infty, 1)$ and $(3, \infty)$. **(2)**

32. Find the intervals in which the function f given by $f(x) = \sin x - \cos x, 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

Delhi 2010

Given function is

$$f(x) = \sin x - \cos x, 0 \le x \le 2\pi$$

On differentiating both sides w.r.t. x, we get

$$f'(x) = \cos x + \sin x \qquad \textbf{(1)}$$

On putting $f'(x) = 0$, we get

$$\cos x + \sin x = 0 \quad \Rightarrow \quad \sin x = -\cos x$$

$$\Rightarrow \quad \frac{\sin x}{\cos x} = -1 \Rightarrow \quad \tan x = -1$$

For $x \in [0, 2\pi]$, $\tan x = \tan \frac{3\pi}{4} \Rightarrow x = \frac{3\pi}{4}$

or $\tan x = \tan \frac{7\pi}{4} \Rightarrow x = \frac{7\pi}{4}$

$$\therefore \quad x = \frac{3\pi}{4}, \frac{7\pi}{4} \qquad \textbf{(2)}$$

[$\because$ tan θ is –1 in 2nd quadrant and 4th quadrant]

Now, we find the intervals in which $f(x)$ is strictly increasing or strictly decreasing.

Interval	Test value	$f'(x) = \cos x + \sin x$	Sign of $f'(x)$
$0 < x < \frac{3\pi}{4}$	At $x = \frac{\pi}{2}$	$f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} + \sin\frac{\pi}{2}$ $= 0 + 1 = 1$	+ve
$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	At $x = \frac{5\pi}{6}$	$f'\left(\frac{5\pi}{6}\right) = \cos\frac{5\pi}{6} + \sin\frac{5\pi}{6}$ $= \cos\left(\pi - \frac{\pi}{6}\right) + \sin\left(\pi - \frac{\pi}{6}\right)$ $= -\cos\frac{\pi}{6} + \sin\frac{\pi}{6}$ $= \frac{-\sqrt{3}}{2} + \frac{1}{2} = \frac{-\sqrt{3}+1}{2}$	–ve
$\frac{7\pi}{4} < x < 2\pi$	At $x = 2\pi$	$f'(2\pi) = \cos 2\pi + \sin 2\pi$ $= \cos(2\pi - 0°) + \sin(2\pi - 0°)$ $= \cos 0° - \sin 0° = 1 - 0 = 1$	+ve

(2)

We know that, a function $f(x)$ is said to be strictly increasing in an interval when $f'(x) > 0$ and it is said to be strictly decreasing when $f'(x) < 0$. So, the given function $f(x)$ is strictly increasing in intervals $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and it is strictly decreasing in the interval $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$. **(1)**

Tangents and Normals

1 Marks Questions

1. Find the slope of tangent to the curve $y = 3x^2 - 6$ at the point on it whose x-coordinate is 2.
All India 2009C

> Firstly, differentiate the given function with respect to x and then determine the value of $\frac{dy}{dx}$ at $x = 2$.

Given, $y = 3x^2 - 6$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 6x$$

At $x = 2$, slope of tangent

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=2} = 6(2) = 12$$

$\therefore$ Required slope = 12 **(1)**

2. Find the slope of tangent to the curve $y = 3x^2 - 4x$ at point whose x-coordinate is 2. Delhi 2009C

Given, $y = 3x^2 - 4x$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 6x - 4$$

At $x = 2$, slope of tangent

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=2} = 6(2) - 4 = 12 - 4 = 8$$

$\therefore$ Required slope = 8 **(1)**

3. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 1$. All India 2008C

Do same as Que. 2. [**Ans.** 8]

4. For the curve $y = 3x^2 + 4x$, find the slope of tangent to the curve at point, where x-coordinate is -2.
Delhi 2008C

Do same as Que. 2. [**Ans.** -8]

4 Marks Questions

5. Find the equations of the tangent and normal to the curves $x = a \sin^3\theta$ and $y = a \cos^3\theta$ at $\theta = \frac{\pi}{4}$.

Delhi 2014

Given, $x = a \sin^3 \theta$

On differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(3 \sin^2 \theta \cos \theta)$$

$$\Rightarrow \quad \frac{dx}{d\theta} = 3a \sin^2 \theta \cos \theta$$

and $y = a \cos^3 \theta$ **(1)**

On differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = -3a \cos^2 \theta \sin \theta$$

Then, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3a \cos^2 \theta \sin \theta}{3a \sin^2 \theta \cos \theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\cot \theta$$ **(1)**

At $\theta = \frac{\pi}{4}$, $\left[\frac{dy}{dx}\right]_{\theta = \frac{\pi}{4}} = -\cot\frac{\pi}{4}$

$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{\theta = \frac{\pi}{4}} = -1$

Also, at $\theta = \frac{\pi}{4}$, $x = a\left(\sin\frac{\pi}{4}\right)^3, y = a\left(\cos\frac{\pi}{4}\right)^3$

$$\Rightarrow \quad x = a\left(\frac{1}{2}\right)^{3/2}, y = a\left(\frac{1}{2}\right)^{3/2}$$

$$\left[\because \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

Now, equation of tangent at the point $\left[\frac{a}{(2)^{3/2}}, \frac{a}{(2)^{3/2}}\right]$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\Rightarrow \quad Y - \frac{a}{(2)^{3/2}} = (-1)\left[X - \frac{a}{2^{3/2}}\right]$$

$$\Rightarrow \quad Y + X = \frac{2a}{(2)^{3/2}}$$

$$\Rightarrow \quad Y + X = \frac{a}{\sqrt{2}} \qquad \textbf{(1)}$$

$$\Rightarrow \quad X + Y - \frac{a}{\sqrt{2}} = 0$$

Also, slope of normal $= \frac{-1}{\text{Slope of tangent}}$

$\Rightarrow \quad$ Slope of normal $= 1$

$\therefore$ Equation of normal at the point $\left[\frac{a}{(2)^{3/2}}, \frac{a}{(2)^{3/2}}\right]$ is

$$Y - \frac{a}{(2)^{3/2}} = (1)\left[X - \frac{a}{2^{3/2}}\right]$$

$$\Rightarrow \quad X - Y = 0 \qquad \textbf{(1)}$$

6. Find the equations of the tangent and normal to the curves $\frac{x^2}{d^2}-\frac{y^2}{b^2}=1$ at the point $(\sqrt{2}a, b)$. **2014**

The equation of the given curve is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

On differentiating both sides w.r.t. x, we get

$$\frac{2x}{a^2}-\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y} \quad \textbf{(1)}$$

$\therefore$ Slope of the tangent at point $(\sqrt{2}a, b)$ is

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(\sqrt{2}a,\, b)}=\frac{\sqrt{2}ab^2}{ba^2}=\frac{\sqrt{2}b}{a} \quad \textbf{(1)}$$

Hence, the equation of the tangent at point $(\sqrt{2}a, b)$ is

$$y-b=\frac{\sqrt{2}b}{a}(x-\sqrt{2}a)$$

$$\Rightarrow \quad a(y-b)=\sqrt{2}\,b(x-\sqrt{2}a)$$

$$\Rightarrow \quad ay-ab=\sqrt{2}\,bx-2ab$$

$$\Rightarrow \quad \sqrt{2}bx-ay-ab=0$$

Now, the slope of the normal at point $(\sqrt{2}a, b)$

$$=\frac{-1}{\text{Slope of tangent}}=\left[\frac{-a^2y}{b^2x}\right]_{(\sqrt{2}a,\, b)}=-\frac{a}{\sqrt{2}b}$$

(1)

Hence, the equation of the normal at point $(\sqrt{2}a, b)$ is

$$(y-b)=-\frac{a}{\sqrt{2}b}(x-\sqrt{2}a)$$

$$\Rightarrow \quad \sqrt{2}\,b(y-b)=-a(x-\sqrt{2}a)$$

$$\Rightarrow \quad \sqrt{2}\,by-\sqrt{2}\,b^2=-ax+\sqrt{2}\,a^2$$

$$\Rightarrow \quad ax+\sqrt{2}by-\sqrt{2}(a^2+b^2)=0 \quad \textbf{(1)}$$

7. Find the points on curve $y = x^3 - 11x + 5$ at which equation of tangent is $y = x - 11$.

Delhi 2012C; HOTS

Firstly, find the slope of given curve and given tangent, then equate them to get value x. Put value of x in given curve to find required points.

Given equation of curve is

$$y = x^3 - 11x + 5 \quad \text{...(i)}$$

Slope of the tangent to the given curve is $\frac{dy}{dx}$.

$$\therefore \quad \frac{dy}{dx} = 3x^2 - 11 \quad \text{...(ii)} \ \textbf{(1)}$$

Also, slope of the tangent $y = x - 11$ is 1.

$$\therefore \quad \frac{dy}{dx} = 1$$

$$\Rightarrow \quad 3x^2 - 11 = 1 \quad \text{[from Eq. (i)]} \ \textbf{(1)}$$

$$\Rightarrow \quad 3x^2 = 12 \Rightarrow x^2 = 4$$

$$\therefore \quad x = \pm 2 \quad \textbf{(1)}$$

Then, from Eq. (i)

When $x = 2$

then $y = (2)^3 - 11(2) + 5$

$= 8 - 22 + 5 = -9$

When $x = -2$, then

$y = (-2)^3 - 11(-2) + 5$

$= -8 + 22 + 5 = 19$

Hence, the required points on the curve are $(2, -9)$ and $(-2, 19)$. **(1)**

8. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which tangent is parallel to X-axis. **Delhi 2011**

We know that, when a tangent is parallel to X-axis, then $\frac{dy}{dx}=0$. So, put $\frac{dy}{dx}=0$ and find value of x from it. Then, put this value of x in the equation of the given curve and find value of y.

Given equation of curve is

$$x^2+y^2-2x-3=0 \quad \text{...(i)}$$

Now, differentiating both sides of Eq. (i) w.r.t. x, we get

$$2x+2y\frac{dy}{dx}-2=0$$

$$\Rightarrow \quad 2y\frac{dy}{dx}=2-2x$$

$$\frac{dy}{dx}=\frac{2-2x}{2y}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{1-x}{y} \quad \textbf{(1)}$$

We know that, when a tangent to the curve is parallel to X-axis, then $\frac{dy}{dx}=0$. **(1)**

On putting $\frac{dy}{dx}=0$, we get

$$1-x=0 \Rightarrow x=1 \quad \textbf{(1)}$$

Now, on putting $x=1$ in Eq. (i), we get

$$1+y^2-2-3=0$$

$$\Rightarrow \quad y^2-4=0 \Rightarrow y^2=4 \Rightarrow y=\pm 2$$

Hence, required points are (1, 2) and (1, –2).

9. Find the points on the curve $y=x^3$ at which the slope of the tangent is equal to y-coordinate of the point. **HOTS; Foreign 2011**

Given, a tangent is equal to y - coordinate of the point, so put $\frac{dy}{dx} = y$ and find value of x from it. Then, put this value of x in the equation of the given curve and find the value of y.

Given equation of curve is $y = x^3$. ...(i)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2$$

$\therefore$ Slope of tangent $= \frac{dy}{dx} = 3x^2$ **(1)**

Now, given that slope of tangent

= y- coordinate of the point

$$\Rightarrow \quad \frac{dy}{dx} = y$$

$$\Rightarrow \quad 3x^2 = y \quad \left[\because \frac{dy}{dx} = 3x^2\right]$$

$$\Rightarrow \quad 3x^2 = x^3 \quad [\because y = x^3]$$

$$\Rightarrow \quad 3x^2 - x^3 = 0 \Rightarrow x^2(3 - x) = 0$$

$$\Rightarrow \quad \text{Either } x^2 = 0 \text{ or } 3 - x = 0$$

$$\therefore \quad x = 0, 3$$ **(1)**

Now, on putting $x = 0, 3$ in Eq. (i), we get

$$y = (0)^3 = 0 \quad [\text{at } x = 0]$$

and $$y = (3)^3 = 27 \quad [\text{at } x = 3]$$ **(1)**

Hence, the required points are (0, 0) and (3, 27). **(1)**

10. Find the equation of tangent to curve $x = \sin 3t$, $y = \cos 2t$ at $t = \frac{\pi}{4}$.

All India 2011C, 2008

We know that, the equation of tangent at the point (x_1, y_1) is $y - y_1 = m(x - x_1)$

where, $$m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)} \quad \text{...(i)}$$ **(1/2)**

Now, given $x = \sin 3t$ ...(ii)

$\therefore \quad \frac{dx}{dt} = 3\cos 3t$ [differentiate w.r.t. t]

and $\quad y = \cos 2t$ ...(iii)

$\therefore \quad \frac{dy}{dt} = -2\sin 2t$ [differentiate w.r.t. t]

Then, $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{-2\sin 2t}{3\cos 3t}$$ **(1)**

On putting $t = \frac{\pi}{4}$, we get

$$m = \left[\frac{dy}{dx}\right]_{t=\frac{\pi}{4}} = \frac{-2\sin\frac{\pi}{2}}{3\cos\frac{3\pi}{4}} = \frac{-2}{-\frac{3}{\sqrt{2}}}$$

$$\left[\because \sin\frac{\pi}{2} = 1 \text{ and } \cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}\right]$$

$\Rightarrow$ $$m = \frac{2\sqrt{2}}{3}$$ **(1)**

Also, to find (x_1, y_1), we put $t = \frac{\pi}{4}$ in Eqs. (ii) and (iii), we get

$$x_1 = \sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and $\quad y_1 = \cos\frac{\pi}{2} = 0$

$\therefore (x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right)$ **(1/2)**

Now, on putting $(x_1, y_1) = \left(\frac{1}{\sqrt{2}}, 0\right)$

and $\quad m = \frac{2\sqrt{2}}{3}$ in Eq. (i), we get

$$y - 0 = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right) \Rightarrow 3y = 2\sqrt{2}x - \frac{2}{3}$$

Hence, required equation of tangent is
$6\sqrt{2}x - 9y - 2 = 0.$ **(1)**

11. Find the equations of tangents to the curve $y = (x^2 - 1)(x - 2)$ at the points, where the curve cuts the X-axis. **All India 2011C**

> The curve cuts the X-axis, so put $y=0$ and get the corresponding values of x. Further, differentiate and determine the slopes at different points. And then use the formula $y-y_1=m(x-x_1)$ to determine the equation of tangent.

Given equation of the curve is

$$y=(x^2-1)(x-2) \quad \text{...(i)}$$

Since, the curve cuts the X-axis, so at that point y- coordinate will be zero.

So, on putting $y=0$, we get

$$(x^2-1)(x-2)=0$$

$$\Rightarrow \quad x^2=1 \text{ or } x=2$$

$$\Rightarrow \quad x=\pm 1 \text{ or } 2$$

$$\Rightarrow \quad x=-1, 1, 2$$

Thus, the given curve cuts the X-axis at points (–1, 0), (1, 0) and (2, 0). **(1)**

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx}=(x^2-1)\cdot 1+(x-2)\cdot 2x$$

[by product rule]

$$\Rightarrow \quad \frac{dy}{dx}=x^2-1+2x^2-4x$$

$$\Rightarrow \quad \frac{dy}{dx}=3x^2-4x-1 \quad \textbf{(1)}$$

12. Find the equation of tangent to the curve $4x^2+9y^2=36$ at the point $(3\cos\theta, 2\sin\theta)$.

Delhi 2011C

Given equation of curve is

$$4x^2 + 9y^2 = 36$$

On differentiating both sides w.r.t. x, we get

$$8x + 18y\frac{dy}{dx} = 0 \Rightarrow 18y\frac{dy}{dx} = -8x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{8x}{18y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-4x}{9y} \quad \text{...(i)} \quad \textbf{(1)}$$

But given that, tangent passes through the point $(3\cos\theta, 2\sin\theta)$.

On putting $x = 3\cos\theta, y = 2\sin\theta$ in Eq. (i), we get

$$\frac{dy}{dx} = \frac{-12\cos\theta}{18\sin\theta}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2\cos\theta}{3\sin\theta}$$

$\therefore$ Slope of the tangent, $m = \dfrac{-2\cos\theta}{3\sin\theta}$ **(1)**

$$\left\{\because m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}\right\}$$

Now, equation of tangent at the point $(3\cos\theta, 2\sin\theta)$ having slope

$$m = -\frac{2\cos\theta}{3\sin\theta} \text{ is}$$

$$y - y_1 = m(x - x_1)$$

$$\Rightarrow \quad y - 2\sin\theta = \frac{-2\cos\theta}{3\sin\theta}(x - 3\cos\theta) \quad \textbf{(1)}$$

$$\Rightarrow 3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$$

$$\Rightarrow 2x\cos\theta + 3y\sin\theta - 6(\sin^2\theta + \cos^2\theta) = 0$$

$$\Rightarrow 2x\cos\theta + 3y\sin\theta - 6 = 0$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

which is the required equation of tangent. **(1)**

13. Find the equation of tangent to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at point $x = 1, y = 0$. **Delhi 2011C**

Given equation of curve is

$$y = x^4 - 6x^3 + 13x^2 - 10x + 5$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10 \quad \textbf{(1)}$$

Slope of a tangent at point (1,0) is

$$m = \left[\frac{dy}{dx}\right]_{x=1} = 4 - 18 + 26 - 10 = 2 \quad \textbf{(1)}$$

∴ Equation of tangent at point (1,0) having slope 2 is **(1)**

$$y - 0 = 2(x - 1)$$

$$\Rightarrow \quad y = 2x - 2$$

Hence, required equation of tangent is $2x - y = 2$. **(1)**

14. Find the values of x for which $f(x) = [x(x-2)]^2$ is an increasing function. Also, find the points on the curve, where the tangent is parallel to X-axis. **Delhi 2010**

(i) Firstly, differentiate the given function with respect to x and put $f'(x) = 0$, then find the value of x and check the interval in which $f'(x) \geq 0$.

(ii) The tangent is parallel to X-axis, i.e. $\frac{dy}{dx} = 0$ get different values of x and put in the given curve to get corresponding values of y.

The given function is $f(x) = [x(x-2)]^2$.

$$\Rightarrow \quad f(x) = (x^2 - 2x)^2 \quad \ldots\text{(i)}$$

We have to find the values of x for which $f(x)$ is an increasing function.

On differentiating both sides of Eq. (i) w.r.t.x, we get

$$f'(x) = 2(x^2 - 2x)\frac{d}{dx}(x^2 - 2x)$$

$$= 2(x^2 - 2x)(2x - 2)$$

On putting $f'(x) = 0$, we get

$$2(x^2 - 2x)(2x - 2) = 0$$

$\Rightarrow \quad 4x(x - 2)(x - 1) = 0$

$\Rightarrow \quad 4x = 0$ or $x - 2 = 0$ or $x - 1 = 0$

$\Rightarrow \quad x = 0, 1, 2$ ...(ii)**(1)**

Now, we find the intervals in which $f(x)$ is an increasing function.

Interval	$f'(x) = 4x(x-1)(x-2)$	Sign of $f'(x)$
$x < 0$	$(-)(-)(-)$	–ve
$0 < x < 1$	$(+)(-)(-)$	+ve
$1 < x < 2$	$(+)(+)(-)$	–ve
$x > 2$	$(+)(+)(+)$	+ve

(1)

From the above table, it is clear that the given function $f(x)$ is an increasing function when $0 < x < 1$ and when $x > 2$. Because at these values of x, $f'(x)$ is positive and we know that, $f(x)$ is said to be an increasing function whenever $f'(x) \geq 0$. **(1)**

Also, we have to find the points on the given curve where the tangent is parallel to X-axis. We know that, when a tangent is parallel to X-axis, then

$$\frac{dy}{dx} = 0$$

$\Rightarrow 2(x^2 - 2x)(2x - 2) = 0$

$\Rightarrow \quad x = 0, 1, 2$

When $x = 0$, then $y = [0(-2)]^2 = 0$

When $x = 1$, then $y = [1(-1)]^2 = 1$

When $x = 2$, then $y = [2(0)]^2 = 0$

Hence, the tangent is parallel to X-axis at the points (0, 0), (1, 1) and (2, 0). **(1)**

15. Find the equation of tangent to the curve $y = \frac{x-7}{x^2 - 5x + 6}$ at the point, where it cuts the X-axis. **All India 2010C, 2010**

Given equation of curve is

$$y = \frac{x-7}{x^2 - 5x + 6} \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 6) \cdot 1 - (x-7)(2x-5)}{(x^2 - 5x + 6)^2}$$

$$\left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left[(x^2 - 5x + 6) - y(x^2 - 5x + 6)(2x-5)\right]}{(x^2 - 5x + 6)^2}$$

$$\left[\because \text{given, } y = \frac{x-7}{x^2 - 5x + 6} \quad \therefore (x-7) = y(x^2 - 5x + 6)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - (2x-5)y}{x^2 - 5x + 6} \quad \text{...(ii)}$$

[dividing numerator and denominator by $x^2 - 5x + 6$] **(1)**

Also, given that given curve cuts X-axis, so its y-coordinate is zero.

$\therefore$ Put $y = 0$ in Eq. (i), we get

$$\frac{x-7}{x^2 - 5x + 6} = 0$$

$$\Rightarrow \quad x = 7 \quad \textbf{(1)}$$

So, curve passes through the point (7, 0).

Now, slope of tangent at (7,0) $= m = \left[\frac{dy}{dx}\right.$

$\lfloor dx \rfloor_{(7, 0)}$

$$= \frac{1-0}{49-35+6} = \frac{1}{20} \quad \textbf{(1)}$$

Hence, the required equation of tangent passing through the point (7, 0) having slope 1/20 is

$$y - 0 = \frac{1}{20}(x - 7) \Rightarrow 20y = x - 7$$

$$\Rightarrow \quad x - 20y = 7 \quad \textbf{(1)}$$

16. Find the equations of the normal to the curve $y = x^3 + 2x + 6$, which are parallel to line $x + 14y + 4 = 0$. **HOTS; Delhi 2010**

> We have to find equation of normal which is parallel to line $x + 14y + 4 = 0$. Here, slope of normal is equal to that of given line. So, first find slope $\frac{-1}{\left(\frac{dy}{dx}\right)}$ of the normal and equate it to slope of the given line. Then, find the required equation.

Given equation of curve is

$$y = x^3 + 2x + 6 \quad ...(i)$$

and the given equation of line is

$$x + 14y + 4 = 0$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2 + 2$$

$$\therefore \text{Slope of normal} = \frac{-1}{\left(\frac{dy}{dx}\right)} = \frac{-1}{3x^2 + 2}$$

Also, slope of the line $x + 14y + 4 = 0$ is $-\frac{1}{14}$.

(1)

[... of the line Ax + By + C = 0 is ... A]

$\left[\because \text{ slope of the line } Ax + By + C = 0 \text{ is} -\frac{A}{B}.\right]$

[∵ we know that, if two lines are parallel, then their slopes are equal]

$\therefore \quad 3x^2 + 2 = 14$

$\Rightarrow \quad 3x^2 = 12 \Rightarrow x^2 = 4$

$\Rightarrow \quad x = \pm 2$ **(1)**

From Eq. (i), when $x = 2$, then

$$y = (2)^3 + 2(2) + 6$$
$$= 8 + 4 + 6 = 18$$

and when $x = -2$, then

$$y = (-2)^3 + 2(-2) + 6$$
$$= -8 - 4 + 6 = -6$$

∴ Normal passes through (2, 18) and (–2, –6).

Also, slope of normal $= \frac{-1}{14}$.

Hence, equation of normal at point (2, 18) is

$$y - 18 = \frac{-1}{14}(x - 2)$$

$\Rightarrow \quad 14y - 252 = -x + 2$

$\Rightarrow \quad x + 14y = 254$ **(1)**

and equation of normal at point (–2, –6) is

$$y + 6 = -\frac{1}{14}(x + 2)$$

$\Rightarrow \quad 14y + 84 = -x - 2$

$\Rightarrow \quad x + 14y = -86$ **(1)**

Hence, the two equations of normal are $x + 14y = 254$ and $x + 14y = -86$.

17. Find the equation of tangent to the curve $x^2 + 3y = 3$, which is parallel to line $y - 4x + 5 = 0$. **Delhi 2009C**

Given equation of the curve is

$$x^2 + 3y = 3 \quad \text{...(i)} \ \textbf{(1/2)}$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$2x + 3\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{2x}{3}$$

$\therefore$ Slope (m) of tangent $= -\frac{2x}{3}$ **(1/2)**

Given equation of the line is

$$y - 4x + 5 = 0 \Rightarrow y = 4x - 5$$

which is of the form $y = mx + c$.

$\therefore$ Slope of the line is $m = 4$.

Now, tangent is parallel to the given line.

$\therefore$ Slope of tangent = Slope of line

$$\Rightarrow -\frac{2x}{3} = 4 \Rightarrow -2x = 12$$

$$\Rightarrow x = -6 \quad \textbf{(1)}$$

On putting $x = -6$ in Eq. (i), we get

$$(-6)^2 + 3y = 3 \Rightarrow 3y = 3 - 36$$

$$\Rightarrow 3y = -33$$

$$y = -11 \quad \textbf{(1)}$$

So, the tangent is passing through point $(-6, -11)$ and it has slope 4.

Hence, the required equation of tangent is

$$y + 11 = 4(x + 6) \Rightarrow y + 11 = 4x + 24$$

$$\Rightarrow 4x - y = -13 \quad \textbf{(1)}$$

18. Find the equation of tangent to the curve $y = \sqrt{3x - 2}$, which is parallel to the line $4x - 2y + 5 = 0$. **Delhi 2009**

Do same as Que. 17. **[Ans.** $48x - 24y = 23$**]**

19. At what points will the tangent to the curve $y = 2x^3 - 15x^2 + 36x - 21$ be parallel to X-axis? Also, find the equations of tangents to the curve.

Given, equation of curve is

$y = 2x^3 - 15x^2 + 36x - 21$ ...(i)

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 6x^2 - 30x + 36 \quad \textbf{(1)}$$

On puttting $\frac{dy}{dx} = 0$, we get

$$6x^2 - 30x + 36 = 0 \Rightarrow 6(x^2 - 5x + 6) = 0$$

$$\Rightarrow x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0$$

$$\Rightarrow x = 2, 3 \quad \textbf{(1)}$$

Now, when $x = 2$, then from Eq. (i), we get

$$y = 2(2)^3 - 15(2)^2 + 36(2) - 21$$

$$\Rightarrow y = 16 - 60 + 72 - 21$$

$$= 88 - 81 = 7$$

Also, when $x = 3$, then from Eq. (i), we get

$$y = 2(3)^3 - 15(3)^2 + 36(3) - 21$$
$$= 54 - 135 + 108 - 21$$
$$\Rightarrow \quad y = 162 - 156 = 6$$

Hence, the tangent passes through the points (2, 7) and (3, 6).

Now, we find the equation of tangents to the given curve.

$\therefore$ Slope (m_1) of tangent at point (2, 7) is

$$m_1 = \left[\frac{dy}{dx}\right]_{(2,7)} = 6(2)^2 - 30(2) + 36$$
$$= 24 - 60 + 36 = 0 \qquad \textbf{(1)}$$

and slope (m_2) of tangent at point (3, 6) is

$$m_2 = \left[\frac{dy}{dx}\right]_{(3,6)} = 6(3)^2 - 30(3) + 36$$
$$= 54 - 90 + 36 = 0$$

$\therefore$ Equation of tangent at point (2, 7) having slope 0 is

$$y - 7 = 0(x - 2) \Rightarrow y - 7 = 0 \Rightarrow y = 7$$

and equation of tangent at point (3, 6) having slope 0 is

$$y - 6 = 0(x - 3)$$
$$\Rightarrow \quad y - 6 = 0 \Rightarrow y = 6 \qquad \textbf{(1)}$$

Hence, equation of tangents are $y = 7$ and $y = 6$.

6 marks Questions

20. Find the equations of the tangent to the curves $y = x^2 - 2x + 7$ which is

(i) parallel to the line $2x - y + 9 = 0$,

(ii) perpendicular to the line $5y - 15x = 13$. **Delhi 2014C**

Given equation of curve is

$$y = x^2 - 2x + 7 \quad \text{...(i)}$$

On differentiating w.r.t. x, we get

$\frac{dy}{dx} = 2x - 2$ **(1)**

(i) The equation of the line is $2x - y + 9 = 0$

$\Rightarrow \quad y = 2x + 9$

which is of the form $y = mx + c$

$\therefore$ Slope of the line is $m = 2$

If a tangent is parallel to the line, then slope of tangent is equal to the slope of the line.

Therefore, $\frac{dy}{dx} = m$

$\Rightarrow \quad 2x - 2 = 2$

$\Rightarrow \quad x = 2$

When $x = 2$, then from Eq. (i), we get

$$y = 2^2 - 2 \times 2 + 7$$

$\Rightarrow \quad y = 7$ **(1)**

The point on the given curve at which tangent is parallel to given line is (2, 7) and the equation of the tangent is

$$y - 7 = 2(x - 2)$$

$\Rightarrow \quad 2x - y + 3 = 0$

Hence, the equation of the tangent line to the given curve which is parallel to line $2x - y + 9 = 0$ is $y - 2x - 3 = 0$. **(1)**

(ii) The equation of the given line is

$5y - 15x = 13$

$\Rightarrow \quad y = \frac{15x + 13}{5} = 3x + \frac{13}{5}$

which is of the from $y = mx + c$.

$\therefore$ Slope of the given line is 3.

If a tangent is perpendicular to the line $5y - 15x = 13$.

Then, the slope of the tangent $= -\frac{1}{3}$

$\therefore 2x-2=\frac{-1}{3} \Rightarrow x=\frac{5}{6}$

When $x=\frac{5}{6}$, then from Eq. (i), we get

$$y=\left(\frac{5}{6}\right)^2-2\left(\frac{5}{6}\right)+7$$

$\Rightarrow \quad y=\frac{217}{36}$ **(1)**

$\therefore$ The point on the given curve at which tangent is perpendicular to given line is $\left(\frac{5}{6}, \frac{217}{36}\right)$ and the equation of the tangent is

$$y-\frac{217}{36}=\frac{-1}{3}\left(x-\frac{5}{6}\right)$$

$\Rightarrow \quad \frac{36y-217}{36}=\frac{-x}{3}+\frac{5}{18}$

$\Rightarrow \quad 12x+36y-227=0$ **(1)**

Hence, the equation of the tangent line to the given curve which is perpendicular to the line $5y-15x=13$ is $36y+12x-227=0.$ **(1)**

21. Find the equation of the normal at a point on the curve $x^2=4y$, which passes through the point (1, 2). Also, find the equation of the corresponding tangent.
Delhi 2013

Given curve is $x^2=4y$

On differentiating both sides w.r.t. x, we get

$2x=4\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{x}{2}$ **(1)**

Let (h, k) be the coordinates of the point of contact of the normal to the curve $x^2=4y$. Then, slope of the tangent at (h, k) is given by

$$\left[\frac{dy}{dx}\right]_{(h,k)}=\frac{h}{2}$$

and slope of the normal at $(h, k)=\frac{-2}{}$

and slope of the normal at $(h, k) = -\frac{2}{h}$

Therefore, the equation of normal at (h, k) is

$$y - k = \frac{-2}{h}(x - h) \quad \text{...(i) } \mathbf{(1)}$$

$$\left[\because \text{ equation of normal in slope form is } y - y_1 = -\frac{1}{m}(x - x_1)\right]$$

Since, it passes through the point (1, 2), so on putting $x = 1$ and $y = 1$, we get

$$2 - k = \frac{-2}{h}(1 - h)$$

$$\Rightarrow \quad k = 2 + \frac{2}{h}(1 - h) \quad \text{...(ii) } \mathbf{(1)}$$

On since, (h, k) also lies on the curve $x^2 = 4y$, so

$$h^2 = 4k \quad \text{...(iii) } \mathbf{(1)}$$

On solving Eqs.(ii) and (iii), we get
$h = 2$ and $k = 1$.

Substituting the values of h and k in Eq.(i), the required equation of normal is

$$y - 1 = \frac{-2}{2}(x - 2) \Rightarrow x + y = 3 \ \mathbf{(1)}$$

Now, equation of tangent at (h, k) is

$$y - k = \frac{h}{2}(x - h)$$

On putting $h = 2$ and $k = 1$, we get

$$y - 1 = \frac{2}{2}(x - 1) \Rightarrow y - 1 = x - 2$$

$$\Rightarrow \quad y = x - 1 \quad \mathbf{(1)}$$

22. Find the equations of tangents to the curve $3x^2 - y^2 = 8$, which passes through the point $\left(\frac{4}{3}, 0\right)$.

HOTS; All India 2013

Firstly, differentiate the given curve with respect to x and determine $\frac{dy}{dx}$. Then, find the equation of tangent at (x_1, y_1) and since it passes through given point (x_0, y_0), so this point will satisfy the tangent and curve also.

Given equation of curve is

$$3x^2 - y^2 = 8 \quad \text{...(i)}$$

On differentiating both sides w.r.t. x, we get

$$6x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{3x}{y} \quad \textbf{(1)}$$

Equation of tangent at point (h, k) is

$$y - k = \left(\frac{dy}{dx}\right)_{(h,\, k)}(x - h)$$

$$\Rightarrow \quad y - k = \frac{3h}{k}(x - h) \quad \text{... (ii)} \ \textbf{(1)}$$

Since, it is passes through the point $\left(\frac{4}{3}, 0\right)$.

$$\therefore \quad 0 - k = \frac{3h}{k}\left(\frac{4}{3} - h\right) \Rightarrow -k^2 = 3h\frac{(4 - 3h)}{3}$$

$$\Rightarrow 3h^2 - k^2 - 4h = 0 \quad \text{...(iii)} \ \textbf{(1)}$$

Also, the point (h, k) satisfy the Eq. (i), so we get

$$3h^2 - k^2 = 8 \quad \text{...(iv)}$$

Now, on solving Eqs. (iii) and (iv), we get

$$4h = 8 \Rightarrow h = 2$$

On putting $h = 2$ in Eq. (iv), we get

$$3(2)^2 - k^2 = 8 \Rightarrow k^2 = 4$$

$$\Rightarrow \quad k = \pm 2 \quad \textbf{(1)}$$

Now, putting the values of h and k in Eq. (ii), we get

$$y - (\pm 2) = \frac{3(2)}{\pm 2}(x - 2)$$

$$\Rightarrow \quad y \mp 2 = \pm 3(x - 2) \Rightarrow y = \pm 3x \mp 6 \pm 2$$

$$\Rightarrow \quad y = \pm 3x \mp 4 \quad \textbf{(1)}$$

It will gives four possible equations but out of them only $y = -3x + 4$ and $y = +3x - 4$ satisfies the point $\left(\frac{4}{3}, 0\right)$. Hence, there are two required equations of tangent. **(1)**

23. For the curve $y = 4x^3 - 2x^5$, find all the points on the curve at which the tangent passes through the origin. **Delhi 2013C**

Given curve is $y = 4x^3 - 2x^5$ ...(i)

Let any point on the curve is (x_1, y_1).

$\therefore \quad y_1 = 4x_1^3 - 2x_1^5$ ...(ii)

On differentiating both sides of Eq. (i), we get

$$\frac{dy}{dx} = 12x^2 - 10x^4 \quad \textbf{(1/2)}$$

Equation of tangent at point (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

$\Rightarrow \quad y - y_1 = [12(x_1)^2 - 10(x_1)^4](x - x_1)$ **(1)**

Since, it passes through the origin.

$\therefore \quad 0 - y_1 = (12x_1^2 - 10x_1^4)(0 - x_1)$

$\Rightarrow \quad y_1 = (12x_1^2 - 10x_1^4)x_1$ ...(iii) **(1/2)**

From Eqs. (ii) and (iii), we get

$(12x_1^2 - 10x_1^4)x_1 = 4x_1^3 - 2x_1^5$

$\Rightarrow 2x_1^3(6 - 5x_1^2) = 2x_1^3(2 - x_1^2)$

$\Rightarrow 2x_1^3(4 - 4x_1^2) = 0 \Rightarrow x_1 = 0$ or $4 - 4x_1^2 = 0$

$\Rightarrow \quad x_1 = 0$ or $x_1 = \pm 1$ **(1)**

On putting the values of $x_1 = 0$, 1 and -1 respectively in Eq. (i), we get

At $x_1 = 0 \Rightarrow y_1 = 0$

At $x_1 = 1$

$\Rightarrow \quad y_1 = 4(1)^3 - 2(1)^5 = 4 - 2 = 2$ **(1)**

and at $x_1 = (-1)$, $y_1 = 4(-1)^3 - 2(-1)^5$

$= 4(-1) - 2(-1) = -4 + 2 = -2$ **(1)**

Hence, all points on the curve at which the tangent passes through origin are (0, 0), (1, 2) and (−1, − 2). **(1)**

24. Find the equations of tangent and normal to the curve $x = 1 - \cos\theta, y = \theta - \sin\theta$ at $\theta = \frac{\pi}{4}$.

All India 2010

Firstly, differentiate the given curve with respect to θ and then determine $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$. Further, use the formula, equation of tangent at (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

and equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{1}{m}(x - x_1).$$

Given curve is $x = 1 - \cos\theta$ and $y = \theta - \sin\theta$.

On differentiating both sides w.r.t. θ, we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(1 - \cos\theta) = \sin\theta$$

and $$\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta - \sin\theta) = 1 - \cos\theta$$

$\therefore$ $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{1 - \cos\theta}{\sin\theta}$$

At $\theta = \frac{\pi}{4}$, $$\left[\frac{dy}{dx}\right]_{\theta = \frac{\pi}{4}} = \frac{1 - \cos\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{1 - \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$$

$$= \sqrt{2} - 1 \quad \textbf{(1)}$$

Also, at $\theta = \frac{\pi}{4}, x_1 = 1 - \cos\frac{\pi}{4} = 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2} - 1}{\sqrt{2}}$

and $$y_1 = \frac{\pi}{4} - \sin\frac{\pi}{4} = \frac{\pi}{4} - \frac{1}{\sqrt{2}} \quad \textbf{(1)}$$

Now, we know that equation of tangent at (x,y) having slope m is given by

$$y - y_1 = m(x - x_1)$$

$$\therefore \; y - \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) = (\sqrt{2} - 1)\left[x - \left(\frac{\sqrt{2} - 1}{\sqrt{2}}\right)\right]$$

$$\Rightarrow \; y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = x(\sqrt{2} - 1) - \frac{(\sqrt{2} - 1)^2}{\sqrt{2}} \quad \textbf{(1)}$$

$$\Rightarrow \quad y - \frac{\pi}{4} + \frac{1}{\sqrt{2}} = x(\sqrt{2}-1) - \frac{(2+1-2\sqrt{2})}{\sqrt{2}}$$

$[\because (a-b)^2 = a^2 + b^2 - 2ab]$

$$\Rightarrow \left(y - \frac{\pi}{4} + \frac{1}{\sqrt{2}}\right) = x(\sqrt{2}-1) - \frac{(3-2\sqrt{2})}{\sqrt{2}}$$

$$\Rightarrow \quad x(\sqrt{2}-1) - y = \frac{3-2\sqrt{2}}{\sqrt{2}} - \frac{\pi}{4} + \frac{1}{\sqrt{2}}$$

Hence, the equation of tangent is

$$x(\sqrt{2}-1) - y = \frac{12 - 8\sqrt{2} - \sqrt{2}\pi + 4}{4\sqrt{2}}$$

$$\Rightarrow x(8 - 4\sqrt{2}) - 4\sqrt{2}y = (16 - \sqrt{2}\pi - 8\sqrt{2}) \quad \textbf{(1)}$$

Also, the equation of normal at (x_1, y_1) having slope $-\frac{1}{m}$ is given by

$$y - y_1 = -\frac{1}{m}(x - x_1)$$

$$\Rightarrow y - \left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}-1}\left(x - \frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y(\sqrt{2}-1) - \left(\frac{\sqrt{2}\pi - 4}{4\sqrt{2}}\right)(\sqrt{2}-1)$$

$$= -x + \frac{\sqrt{2}-1}{\sqrt{2}} \quad \textbf{(1)}$$

$$\Rightarrow \quad y(\sqrt{2}-1) - \left(\frac{2\pi - \sqrt{2}\pi - 4\sqrt{2} + 4}{4\sqrt{2}}\right)$$

$$= \frac{-\sqrt{2}x + \sqrt{2} - 1}{\sqrt{2}}$$

$$\Rightarrow \quad 4\sqrt{2}y(\sqrt{2}-1) - 2\pi + \sqrt{2}\pi + 4\sqrt{2} - 4$$

$$= -4\sqrt{2}x + 4\sqrt{2} - 4$$

$$\Rightarrow \quad 4\sqrt{2}x + 4\sqrt{2}y(\sqrt{2}-1) = 2\pi - \sqrt{2}\pi$$

$$\Rightarrow \quad 4\sqrt{2}x + y(8 - 4\sqrt{2}) = 2\pi - \sqrt{2}\pi$$

$$\Rightarrow \quad 4\sqrt{2}x + (8 - 4\sqrt{2})y = \pi(2 - \sqrt{2}) \quad \textbf{(1)}$$

Maximaand Minima

Previous Year Examination Questions

4 Marks Questions

1. The sides of an equilateral triangle are increasing at the rate of 2cm/s. Find the rate at which the area increases, when the side is 10 cm. **All India 2014C**

Let the side of triangle be a.

$$\frac{da}{dt} = 2 \text{ cm/s} \qquad \text{[given]} \quad \textbf{(1)}$$

Now, area of equilateral triangle having side a is given by

$$A = \frac{\sqrt{3}a^2}{4} \qquad \textbf{(1)}$$

On differentiating w.r.t.*t*, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \cdot (2a) \frac{da}{dt} \qquad \textbf{(1)}$$

On putting $\frac{da}{dt} = 2$ cm/s and $a = 10$ cm, we get

$$\frac{dA}{dt} = \frac{\sqrt{3}}{4} \times 2 \times 10 \times 2 = 10\sqrt{3} \text{ cm}^2/\text{s} \qquad \textbf{(1)}$$

2. The sum of the perimeters of a circle and square is *k*, where *k* is some constant. Prove that the sum of their areas is least, when the side of the square is double the radius of the circle. **Delhi 2014C; All India 2008**

Let *r* be the radius of circle and x be the side of a square. Then, given that

Perimeter of square + Circumference of circle

$$= k \text{ (constant)} \qquad \textbf{(1)}$$

i.e. $\quad 4x + 2\pi r = k$

$$\Rightarrow \quad x = \frac{k - 2\pi r}{4} \quad \text{...(i)} \; \textbf{(1)}$$

Let A denotes the sum of their areas.

$$\therefore \quad A = x^2 + \pi r^2 \quad \text{...(ii)}$$

$$\left[\begin{array}{l} \because \text{ area of a square} = (\text{Side})^2 \\ \text{and area of circle } = \pi r^2 \end{array}\right]$$

On putting the value of x from Eq. (i) in Eq.(ii), we get

$$A = \left(\frac{k - 2\pi r}{4}\right)^2 + \pi r^2$$

On differentiating w.r.t. r, we get

$$\frac{dA}{dr} = 2\left(\frac{k - 2\pi r}{4}\right)\left(-\frac{2\pi}{4}\right) + 2\pi r$$

$$= -\frac{\pi}{4}(k - 2\pi r) + 2\pi r \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dA}{dr} = 0$

$$\Rightarrow \quad -\frac{\pi}{4}(k - 2\pi r) + 2\pi r = 0$$

$$\Rightarrow \quad -\frac{\pi}{4}k + \frac{\pi^2 r}{2} + 2\pi r = 0$$

$$\Rightarrow \quad -\frac{r\pi}{2}(\pi + 4) = \frac{\pi}{4}k$$

$$\Rightarrow \quad r = \frac{k}{2\pi + 8} \qquad \text{...(iii)}$$

Now, $\frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{dA}{dr}\right) = \frac{d}{dr}\left[2\pi r - \frac{\pi}{4}(k - 2\pi r)\right]$

$$= 2\pi + \frac{2\pi^2}{4} = 2\pi + \frac{\pi^2}{2} > 0$$

$\because \quad \frac{d^2A}{dr^2} > 0 \Rightarrow A$ is minimum.

From Eq. (iii), we get

$$r = \frac{k}{2\pi + 8}$$

$\Rightarrow \quad 2r\pi + 8r = k$

$\Rightarrow \quad 2r\pi + 8r = 4x + 2\pi r \qquad [\because k = 4x + 2\pi r]$

$\Rightarrow \quad 8r = 4x$ or $x = 2r$

i.e. Side of square = Double the radius of circle

Hence, sum of area of a circle and a square is least, when side of square is equal to diameter of circle or double the radius of circle. **(1)**

6 Marks Questions

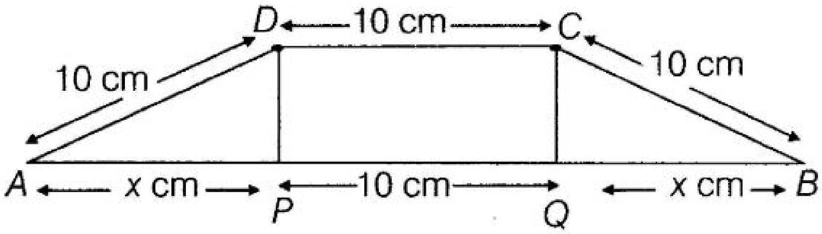
3. If the length of three sides of a trapezium other than the base are each equal to 10 cm, then find the area of the trapezium, when it is maximum. **All India 2014C, 2010; Delhi 2013C**

Let $ABCD$ be the given trapezium in which $AD = BC = CD = 10$ cm.

Let $\quad AP = x \text{ cm}$

$\because \quad \Delta APD \cong \Delta BQC$

$\therefore \quad QB = x \text{ cm}$ **(1)**



In ΔAPD,

$$DP = \sqrt{10^2 - x^2} \text{ [by Pythagoras theorem]}$$

Now, area of trapezium,

$$A = \frac{1}{2} \times \text{(Sum of parallel sides)} \times \text{Height}$$

$$= \frac{1}{2} \times (2x + 10 + 10) \times \sqrt{100 - x^2}$$

$$= (x + 10)\sqrt{100 - x^2} \quad \text{... (i)} \textbf{(1)}$$

On differentiating both sides of Eq. (i) w.r.t. x, we get

$$\frac{dA}{dx} = (x + 10)\frac{(-2x)}{2\sqrt{100 - x^2}} + \sqrt{100 - x^2}$$

$$= \frac{-x^2 - 10x + 100 - x^2}{\sqrt{100 - x^2}}$$

$$= \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}} \quad \text{...(ii)} \textbf{ (1)}$$

For maximum, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}} = 0$$

$$\Rightarrow \quad 2(x + 10)(x - 5) = 0 \Rightarrow x = 5 \text{ or } -10 \quad \textbf{(1)}$$

Since, x represents distance, it cannot be negative. So, we take $x = 5$.

On differentiating both sides of Eq. (ii) w.r.t. x, we get

$$\left[\sqrt{100 - x^2}(-4x - 10) \ldots \left(\frac{-2x}{\ldots} \right) \right]$$

$$\frac{d^2A}{dx^2} = \frac{\left[-(-2x^2 - 10x + 100)\left(\frac{-2x}{2\sqrt{100 - x^2}}\right)\right]}{(\sqrt{100 - x^2})^2}$$

[by quotient rule]

$$= \frac{\begin{bmatrix}(100 - x^2)(-4x - 10) \\ -(-2x^2 - 10x + 100)(-x)\end{bmatrix}}{(100 - x^2)^{3/2}}$$

$$= \frac{\begin{bmatrix}-400x - 1000 + 4x^3 + 10x^2 \\ +(-2x^3 - 10x^2 + 100x)\end{bmatrix}}{(100 - x^2)^{3/2}}$$

$$= \frac{2x^3 - 300x - 1000}{(100 - x^2)^{3/2}} \quad \textbf{(1)}$$

At $x = 5$, $\frac{d^2A}{dx^2} = \frac{2(5)^3 - 300(5) - 1000}{[100 - (5)^2]^{3/2}}$

$$= \frac{250 - 1500 - 1000}{(100 - 25)^{3/2}} = \frac{-2250}{75\sqrt{75}} < 0$$

Thus, area of trapezium is maximum at $x = 5$ and maximum value is

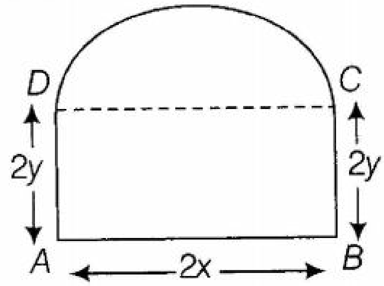
$A_{max} = (5 + 10)\sqrt{100 - (5)^2}$ [put $x = 5$ in Eq. (i)]

$= 15\sqrt{100 - 25} = 15\sqrt{75} = 75\sqrt{3}$ cm^2 **(1)**

4. A window is of the form of a semi-circle with a rectangle on its diameter. The total perimeter of the window is 10m. Find the dimensions of the window to admit maximum light through the whole opening.

Foreign 2014; All India 2011

Let the dimensions of the window be $2x$ and $2y$, i.e. $AB = 2x$ and $BC = 2y$. Again, let P denotes the perimeter of the window and A denotes its area.



Given, $P = 10$ m

To find dimensions of the window, so that maximum light pass through the whole opening.

From the figure, we can see that perimeter of the window is given by

$$P = 2x + 2y + 2y + \pi x$$

[$\because$ in the semi-circle, $2x$ is the diameter, so radius is x]

$$\Rightarrow \quad P = 2x + 4y + \pi x$$

$$\Rightarrow \quad 2x + 4y + \pi x = 10 \quad [\because P = 10 \text{ m}]$$

$$\Rightarrow \quad y = \frac{10 - \pi x - 2x}{4} \quad \text{...(i)} \ \mathbf{(1)}$$

Also, area of the whole window is given by

$$A = (2x)(2y) + \frac{\pi x^2}{2}$$

[$\because$ area of window = area of rectangle + area of semi-circle]

$$\Rightarrow \quad A = 4xy + \frac{\pi x^2}{2} \quad \text{...(ii)}$$

5.

On putting value of y from Eq. (i) in Eq. (ii), we get

$$A = 4x\left(\frac{10 - \pi x - 2x}{4}\right) + \frac{\pi x^2}{2}$$

$$\Rightarrow \quad A = 10x - \pi x^2 - 2x^2 + \frac{\pi x^2}{2}$$

$$\Rightarrow \quad A = 10x - 2x^2 - \frac{\pi x^2}{2} \quad \mathbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dA}{dx} = 10 - 4x - \frac{2\pi x}{2}$$

$$= 10 - 4x - \pi x \quad \mathbf{(1)}$$

For maxima and minima, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad 10 - 4x - \pi x = 0$$

$$\Rightarrow \quad x = \frac{10}{\pi + 4} \qquad \textbf{(1)}$$

On putting value of x in Eq. (i), we get

$$y = \frac{10 - \pi\left(\frac{10}{\pi + 4}\right) - 2\left(\frac{10}{\pi + 4}\right)}{4}$$

$$\Rightarrow \quad y = \frac{\left(10 - \frac{10\pi}{\pi + 4} - \frac{20}{\pi + 4}\right)}{4}$$

$$\Rightarrow \quad y = \frac{20}{4(\pi + 4)} = \frac{5}{\pi + 4} \qquad \textbf{(1)}$$

$$\therefore \quad x = \frac{10}{\pi + 4} \text{ and } y = \frac{5}{\pi + 4}$$

Now, $\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{dA}{dx}\right) = \frac{d}{dx}(10 - 4x - \pi x)$

$$= -4 - \pi < 0$$

$\therefore$ A is maximum.

Hence, maximum light passes through the window.

And dimensions of window are

$$2x = \frac{20}{\pi + 4} \text{ and } 2y = \frac{10}{\pi + 4} \qquad \textbf{(1)}$$

5. Find the point p on the curve $y^2 = 4ax$, which is nearest to the point $(11\,a, 0)$.

All India 2014C

Let the point on $y^2 = 4ax$ be (x_1, y_1). Then, $y_1^2 = 4ax_1$. ...(i)

Distance between (x_1, y_1) and $(11a, 0)$ is given by

$$D = \sqrt{(x_1 - 11a)^2 + (y_1 - 0)^2}$$

$$= \sqrt{(x_1 - 11a)^2 + y_1^2}$$

$$= \sqrt{(x_1 - 11a)^2 + 4ax_1} \text{ [from Eq. (i)] } \mathbf{(1)}$$

On differentiating both sides w.r.t. x_1, we get

$$\frac{dD}{dx_1} = \frac{1}{2\sqrt{(x_1 - 11a)^2 + 4ax_1}} \cdot [2(x_1 - 11a) + 4ax_1] \ \mathbf{(1)}$$

$$\Rightarrow \quad \frac{dD}{dx_1} = \frac{2x_1 - 22a + 4a}{2\sqrt{(x_1 - 11a)^2 + 4ax_1}} = 0$$

$$\Rightarrow \quad \frac{dD}{dx_1} = \frac{x_1 - 9a}{\sqrt{(x_1 - 11a^2) + 49x_1}}$$

$\Rightarrow$ Put $\quad \dfrac{dD}{dx_1}=0 \quad \Rightarrow x_1-9a=0$

$\Rightarrow \quad x_1 = 9a$

If $x_1 = 9a$, then $y_1^2 = 36a^2$

$\Rightarrow \quad y_1 = \pm 6a$ **(1)**

Hence, required points are $(9a, 6a)$ and $(9a,-6a)$.

Now, $\quad \dfrac{d^2D}{dx_1^2} = \dfrac{d}{dx_1}\left[\dfrac{dD}{dx_1}\right]$ **(1)**

$$= \frac{d}{dx_1}\left(\frac{x_1-9a}{\sqrt{(x_1-11a)^2+49x_1}}\right)$$

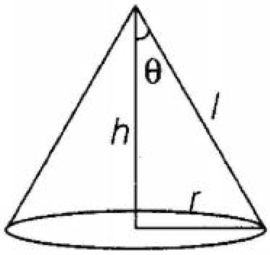
$$= \frac{\sqrt{(x_1-11a)^2+4ax_1}-(x_1-9a)\cdot\dfrac{1\cdot[2(x_1-11a)+4a]}{2\sqrt{(x_1-11a)^2+4ax_1}}}{(x_1-11a)^2+4ax_1}$$

At $(9a, 6a)$, $\dfrac{d^2D}{dx_1^2}>0$ **(1)**

So, at $(9a, 6a)$, D is minimum.

Hence, required point is $(9a,6a)$. **(1)**

Let r be the radius of the base, h be the height, V be the volume, S be the surface area of the cone and θ be the semi-vertical angle.



(1)

Then, $V = \frac{1}{3}\pi r^2 h$

$\Rightarrow \quad 3V = \pi r^2 h$

$\Rightarrow \quad 9V^2 = \pi^2 r^4 h^2$ [squaring on both sides]

$\Rightarrow \quad h^2 = \frac{9V^2}{\pi^2 r^4}$ ...(i) **(1)**

and curved surface area, $S = \pi r l$

$\Rightarrow \quad S = \pi r\sqrt{r^2 + h^2}$ $[\because l = \sqrt{h^2 + r^2}]$

$\Rightarrow \quad S^2 = \pi^2 r^2 (r^2 + h^2)$

[squaring on both sides]

$\Rightarrow \quad S^2 = \pi^2 r^2\left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$ [from Eq. (i)]

$\Rightarrow \quad S^2 = \frac{9V^2}{r^2} + \pi^2 r^4$ ...(ii)

When S is least, then S^2 is also least. **(1)**

Now, $\frac{d}{dr}(S^2) = -\frac{18V^2}{r^3} + 4\pi^2 r^3$ ...(iii)

For maxima or minima, put $\frac{d}{dr}(S^2) = 0$ **(1)**

$\Rightarrow \quad -\frac{18V^2}{r^3} + 4\pi^2 r^3 = 0 \Rightarrow 18V^2 = 4\pi^2 r^6$

$\Rightarrow \quad 9V^2 = 2\pi^2 r^6$ ...(iv) **(1)**

Again, on differentiating Eq. (iii) w.r.t.r, we get

$$\frac{d^2}{dr^2}(S^2) = \frac{54V^2}{r^4} + 12\pi^2 r^2$$

At $\quad 9V^2 = 2\pi^2 r^6$,

$$\frac{d^2}{dr^2}(S^2) = \frac{54}{r^4}\left(\frac{2\pi^2 r^6}{9}\right) + 12\pi^2 r^2$$

$$= \frac{12\pi^2 r^6}{r^4} + 12\pi^2 r^2 = 24\pi^2 r^2 > 0$$

So, S^2 and S is minimum, when

$$9V^2 = 2\pi^2 r^6$$ **(1)**

On putting $9V^2 = 2\pi^2 r^6$ in Eq. (i), we get

$$2\pi^2 r^6 = \pi^2 r^4 h^2$$

$\Rightarrow \quad 2r^2 = h^2 \Rightarrow h = \sqrt{2}\, r$

$\Rightarrow \quad \frac{h}{r} = \sqrt{2} \Rightarrow \cot\theta = \sqrt{2}$

[from the figure, $\cot\theta = \frac{h}{r}$]

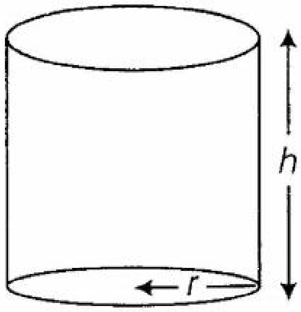
$\Rightarrow \quad \theta = \cot^{-1}\sqrt{2}$

Hence, the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. **(1)**

Hence proved.

7. Of all the closed right circular cylindrical cans of volume $128\pi\ \text{cm}^3$, find the dimensions of the can which has minimum surface area.
Delhi 2014

Let r cm be the radius of base and h cm be the height of the cylindrical can. Let its volume be V and S be its total surface area. Then,



Can **(1)**

$$V = 128\pi \text{ cm}^3 \quad \text{[given]}$$

$$\Rightarrow \quad \pi r^2 h = 128\pi$$

$$\Rightarrow \quad h = \frac{128}{r^2} \quad \text{...(i)}$$

Also, $$S = 2\pi r^2 + 2\pi rh \quad \text{...(ii)}$$

$$\Rightarrow \quad S = 2\pi r^2 + 2\pi r\left(\frac{128}{r^2}\right) \text{ [using Eq.(i)]}$$

(1)

$$\Rightarrow \quad S = 2\pi r^2 + \frac{256\pi}{r} \quad \text{...(iii)}$$

On differentiating Eq. (iii) w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2} \quad \text{...(iv)}$$ **(1)**

For maxima or minima, put $\frac{dS}{dr} = 0$.

$$\Rightarrow \quad 4\pi r = \frac{256\pi}{r^2}$$

$$\Rightarrow \quad r^3 = \frac{256}{4}$$

$$\Rightarrow \quad r^3 = 64$$

Taking cube root on both sides, we get

$$r = (64)^{1/3}$$

$$\Rightarrow \quad r = 4 \text{ cm}$$ **(1)**

Again,on differentiating Eq. (iv) w.r.t. r, we get

$$\frac{d^2S}{dr^2} = 4\pi + \frac{512\pi}{r^3}$$

At $r = 4$,

$$\frac{d^2S}{dr^2} = \frac{512\pi}{64} + 4\pi$$

$$= 8\pi + 4\pi = 12\pi > 0 \qquad \textbf{(1)}$$

Thus, $\frac{d^2s}{dr^2} > 0$ at $r = 4$, so the surface area is minimum, when the radius of cylinder is 4 cm.

On putting value of r in Eq. (i), we get

$$h = \frac{128}{(4)^2}$$

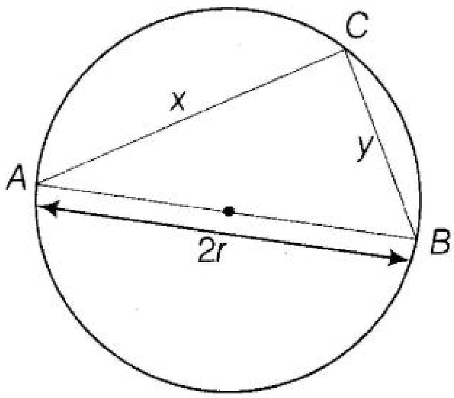
$$= \frac{128}{16} = 8 \text{ cm}$$

Hence, for the minimum surface area of can, the dimensions of the can are $r = 4$ cm and $h = 8$ cm. **(1)**

8. AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles. **All India 2014C**

Let the side of ΔABC be x and y and r be the radius of circle.

Also, $\angle C = 90°$ [$\because$ angle made in semi-circle]



(1)

In the ΔABC, we have

$$(AB)^2 = (AC)^2 + (BC)^2$$

$$\Rightarrow \quad (2r)^2 = (x)^2 + (y)^2$$

$$\Rightarrow \quad 4r^2 = x^2 + y^2 \qquad \text{...(i)}\textbf{(1)}$$

Area of ΔABC $(A) = \frac{1}{2} x \cdot y$

On squaring both sides, we get

$$A^2 = \frac{1}{4} x^2 y^2$$

Let $A^2 = S$

Then, $S = \frac{1}{4} x^2 y^2$

From the Eq. (i), substituting the value of y^2, we get

$$S = \frac{1}{4} x^2 (4r^2 - x^2)$$

$$\Rightarrow \quad S = \frac{1}{4} (4x^2 r^2 - x^4) \quad \textbf{(1)}$$

On differentiating w.r.t.x, we get

$$\frac{dS}{dx} = \frac{1}{4} (8r^2 x - 4x^3)$$

For maximum and minimum, put $\frac{dS}{dx} = 0$.

$$\Rightarrow \quad 0 = \frac{1}{4} (8r^2 x - 4x^3)$$

$$\Rightarrow \quad 8r^2 x = 4x^3 \Rightarrow 8r^2 = 4x^2$$

$$\Rightarrow \quad x^2 = 2r^2 \Rightarrow x = \sqrt{2} r$$

Then, from the Eq.(i), we get

$$y^2 = 4r^2 - 2r^2 = 2r^2$$

$$\Rightarrow \quad y = \sqrt{2} r \quad \textbf{(1)}$$

i.e. $x = y$, so triangle is isosceles.

Also, $\frac{d^2S}{dx^2} = \frac{d}{dx}\left[\frac{1}{4}(8r^2x - 4x^3)\right]$ **(1)**

$$= \frac{1}{4}[8r^2 - 12x^2] = 2r^2 - 3x^2$$

At $x = \sqrt{2}r, \frac{d^2S}{dx^2} = 2r^2 - 3x^2 < 0$

$\Rightarrow$ Area is maximum.

Hence, area is maximum, when triangle is isosceles. **(1)**

9. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also, show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere. **All India 2014**

Let R be the radius and h be the height of the cone, which inscribed in a sphere of radius r.

$\therefore \qquad OA = h - r$

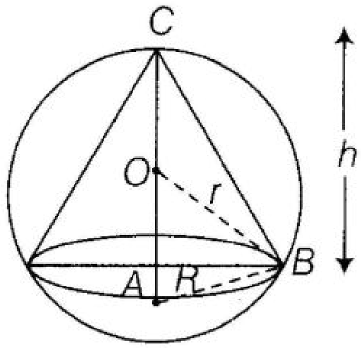
In ΔOAB, by Pythagoras theorem, we have

$$r^2 = R^2 + (h - r)^2$$

$$\Rightarrow \qquad r^2 = R^2 + h^2 + r^2 - 2rh$$

$$\Rightarrow \qquad R^2 = 2rh - h^2 \qquad \text{...(i)} \mathbf{(1)}$$

The volume of sphere $= \frac{4}{3}\pi r^3$



and the volume V of the cone,

$$V = \frac{1}{3}\pi R^2 h$$

$$\Rightarrow \qquad V = \frac{1}{3}\pi h(2rh - h^2) \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \qquad V = \frac{1}{3}\pi(2rh^2 - h^3) \qquad \text{...(ii)}$$

On differentiating Eq. (ii) w.r.t. h, we get

$$\frac{dV}{dh} = \frac{1}{3}\pi(4rh - 3h^2) \qquad \text{...(iii)} \mathbf{(1)}$$

For maximum or minimum, put $\frac{dV}{dh} = 0$.

$$\Rightarrow \frac{1}{3}\pi(4rh - 3h^2) = 0$$

$$\Rightarrow \qquad 4rh = 3h^2 \Rightarrow 4r = 3h$$

$$\Rightarrow \qquad h = \frac{4r}{3} \qquad [\because h \neq 0] \ \mathbf{(1)}$$

Again, on differentiating Eq. (iii) w.r.t. h, we get

$$\frac{d^2V}{dh^2} = \frac{1}{3}\pi(4r - 6h)$$

At $h = \frac{4r}{3}$, $\left[\frac{d^2V}{dh^2}\right]_{h=\frac{4r}{3}} = \frac{1}{3}\pi\left(4r - 6 \times \frac{4r}{3}\right)$

$$= \frac{\pi}{3}(4r - 8r) = -\frac{4r\pi}{3} < 0$$

$\Rightarrow V$ is maximum at $h = \frac{4r}{3}$. **(1)**

On substituting the value of h in Eq. (ii), we get

$$V = \frac{1}{3}\pi\left[2r\left(\frac{4r}{3}\right)^2 - \left(\frac{4r}{3}\right)^3\right]$$

$$= \frac{\pi}{3}\left[\frac{32}{9}r^3 - \frac{64}{27}r^3\right] = \frac{\pi}{3}r^3\left[\frac{32}{9} - \frac{64}{27}\right]$$

$$= \frac{\pi}{3}r^3\left[\frac{96-64}{27}\right] = \frac{\pi}{3}r^3\left(\frac{32}{27}\right)$$

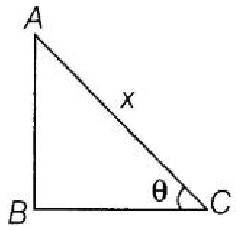
$= \frac{8}{27} \times \left(\frac{4}{3}\pi r^3\right) = \frac{8}{27} \times$ (Volume of sphere) **(1)**

Hence, maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

10. If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, then show that the area of the triangle is maximum, when the angle between them is $60°$. **All India 2014**

Let ABC be a right angled triangle.

Given, $AC + BC = \text{constant} = k$ ...(i)



(1)

Let $\angle ACB = \theta$ and $AC = x$.

Then, $BC = x\cos\theta$ and $AB = x\sin\theta$

Let y be the area of ΔABC.

Then, $y = \frac{1}{2} BC \cdot AB = \frac{1}{2} x\cos\theta \cdot x\sin\theta$

$$= \frac{1}{2} x^2 \sin\theta\cos\theta \quad \text{...(ii)}$$

From Eq. (i), $x + x\cos\theta = k$

$$\Rightarrow \quad x = \frac{k}{1+\cos\theta} \quad \text{...(iii)} \textbf{(1)}$$

On putting the value of x in Eq. (ii), we get

$$y = \frac{k^2}{2} \frac{\sin\theta\cos\theta}{(1+\cos\theta)^2}$$

On differentiating w.r.t. θ, we get

$$\frac{dy}{dx} = \frac{k^2}{2} \frac{\begin{bmatrix}(1+\cos\theta)^2(\cos^2\theta - \sin^2\theta) \\ -\sin\theta\cos\theta\, 2(1+\cos\theta)(-\sin\theta)\end{bmatrix}}{(1+\cos\theta)^4}$$

$$= \frac{k^2}{2} \frac{\begin{bmatrix}(1+\cos\theta)[(1+\cos\theta)(\cos^2\theta - \sin^2\theta) \\ +2\sin^2\theta\cos\theta]\end{bmatrix}}{(1+\cos\theta)^4}$$

(1)

$$= \frac{k^2}{2} \frac{\begin{bmatrix}\cos^2\theta - \sin^2\theta + \cos^3\theta - \cos\theta\sin^2\theta \\ +2\cos\theta\sin^2\theta\end{bmatrix}}{(1+\cos\theta)^3}$$

$$= \frac{k^2(2\cos^2\theta - 1 + \cos^3\theta + \cos\theta\sin^2\theta)}{2(1+\cos\theta)^3}$$

$[\because \sin^2\theta = 1 - \cos^2\theta]$

$$= \frac{k^2[2\cos^2\theta - 1 + \cos\theta(\cos^2\theta + \sin^2\theta)]}{2(1+\cos\theta)^3}$$

$$= \frac{k^2}{2(1+\cos\theta)^3}(2\cos^2\theta + \cos\theta - 1)$$

$[\because \cos^2\theta + \sin^2\theta = 1]$ **(1)**

Since, $0 < \theta < \frac{\pi}{2} \Rightarrow \frac{k^2}{2(1+\cos\theta)^3} > 0$

$\therefore$ Sign of $\frac{dy}{dx}$ will be depend on $2\cos^2\theta + \cos\theta - 1$.

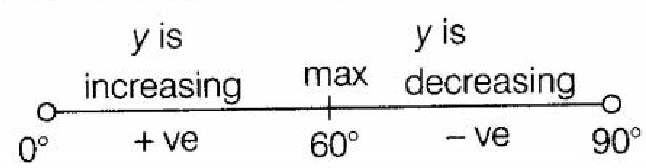
Now, $\quad 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2} \quad [\because \cos\theta \neq -1]$

$\Rightarrow \quad \theta = 60° \quad [\because 0 < \theta < 90°]$ **(1)**

Then, sign scheme for $\frac{dy}{dx}$, i.e. for $(2\cos^2\theta + \cos\theta - 1)$ is



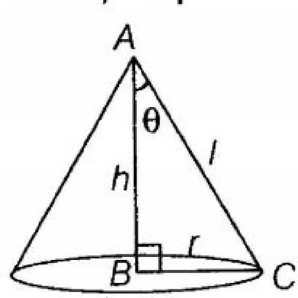
Thus, y has maximum value, when $\theta = 60°$.**(1)**

11. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1} 1/\sqrt{3}$. **Delhi 2014**

Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left(0, \frac{\pi}{2}\right)$.

Let r, h and l be the radius, height and the slant height of the cone, respectively.



(1)

The slant height of the cone is given, i.e. consider as constant.

Now, in ΔABC, $r = l\sin\theta$ and $h = l\cos\theta$

Now, in ΔABC, $r = l \sin\theta$ and $h = l\cos\theta$

Let V be the volume of the cone.

Then, $V = \frac{\pi}{3} r^2 h$

$$\Rightarrow \quad V = \frac{1}{3}\pi (l^2 \sin^2\theta)(l \cos\theta)$$

$$\Rightarrow \quad V = \frac{1}{3}\pi l^3 \sin^2\theta \cos\theta \qquad \textbf{(1)}$$

On differentiating w.r.t. θ two times, we get

$$\frac{dV}{d\theta} = \frac{l^3\pi}{3}[\sin^2\theta(-\sin\theta) + \cos\theta(2\sin\theta\cos\theta)]$$

$$= \frac{l^3\pi}{3}(-\sin^3\theta + 2\sin\theta\cos^2\theta)$$

and $$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta)$$

$$\Rightarrow \frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 7\sin^2\theta\cos\theta) \qquad \textbf{(1)}$$

For maxima or minima, put $\frac{dV}{d\theta} = 0$.

$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta \quad \Rightarrow \quad \tan^2\theta = 2$

$\Rightarrow \tan\theta = \sqrt{2} \quad \Rightarrow \quad \theta = \tan^{-1}\sqrt{2}$ **(1)**

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$

$$\Rightarrow \quad \sin^2\theta = 2\cos^2\theta$$

Then, we have

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}(2\cos^3\theta - 14\cos^3\theta)$$

$$= -4\pi l^3\cos^3\theta < 0, \text{ for } \theta \in \left(0, \frac{\pi}{2}\right) \qquad \textbf{(1)}$$

$\therefore V$ is maximum, when $\theta = \tan^{-1}\sqrt{2}$ or

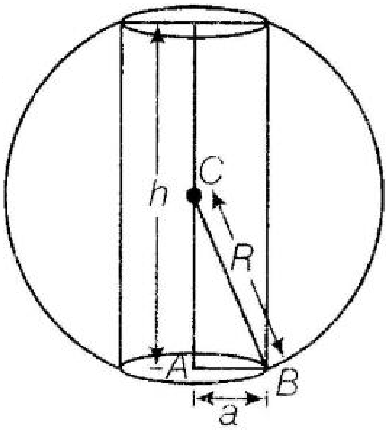
$$\theta = \cos^{-1}\frac{1}{\sqrt{3}}. \qquad \left[\because \cos\theta = \frac{1}{\sqrt{3}}\right]$$

Hence, for given slant height, the semi-vertical angle of the cone of maximum volume is $\cos^{-1}\frac{1}{\sqrt{3}}$. **(1)**

12. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also, find the maximum volume.

All India 2014,2012C,2011; Delhi 2013

Let h be the height and a be the radius of base of cylinder inscribed in the given sphere of radius (R).



In ΔABC,

$$AB^2 + AC^2 = BC^2 \text{ [by Pythagoras theorem]}$$

$$\Rightarrow a^2 + \left(\frac{h}{2}\right)^2 = R^2 \Rightarrow a^2 = R^2 - \frac{h^2}{4} \quad \textbf{(1)}$$

Volume of cylinder, $V = \pi a^2 h$

$$= \pi h\left(R^2 - \frac{h^2}{4}\right) = \frac{\pi}{4}(4R^2 h - h^3) \quad \textbf{(1)}$$

On differentiating both sides two times w.r.t. h, we get

$$\frac{dV}{dh} = \frac{\pi}{4}(4R^2 - 3h^2)$$

and $$\frac{d^2V}{dh^2} = \frac{\pi}{4}(-6h) = -\frac{3\pi h}{2} \quad \text{...(i)} \ \textbf{(1)}$$

For maxima or minima, put $\frac{dV}{dh} = 0$

$$\Rightarrow \quad \frac{\pi}{4}(4R^2 - 3h^2) = 0$$

$$\Rightarrow \quad h^2 = \frac{4}{3}R^2$$

$$\Rightarrow \quad h = \frac{2}{\sqrt{3}}R \quad \text{(1)}$$

[$\because$ height is always positive, so we do not take '–' sign]

On substituting value of h in Eq. (i), we get

$$\frac{d^2V}{dh^2} = \frac{-3\pi}{2} \cdot \frac{2}{\sqrt{3}}R = -\sqrt{3}\pi R < 0 \quad \text{(1)}$$

$\Rightarrow$ V is maximum.

Hence, required height of cylinder is $\frac{2R}{\sqrt{3}}$.

Now, maximum volume of cylinder

$$= \pi h\left(R^2 - \frac{h^2}{4}\right) = \pi \frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4} \cdot \frac{4}{3}R^2\right)$$

$$\left[\text{put } h = \frac{2}{\sqrt{3}}R\right]$$

$$= \frac{2\pi R}{\sqrt{3}} \frac{(3R^2 - R^2)}{3} = \frac{4\pi R^3}{3\sqrt{3}} \text{ cu units} \quad \text{(1)}$$

Hence proved.

13. Show that a cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base. **Foreign 2014; Delhi 2011C, 2009**

Let r be the radius, h be the height, V be the volume and S be the total surface area of a right circular cylinder which is open at the top. Now, given that

$$V = \pi r^2 h \quad \Rightarrow \quad h = \frac{V}{\pi r^2} \qquad \text{...(i)} \ \textbf{(1)}$$

Also, we know that, total surface area S is given by

$$S = 2\pi r h + \pi r^2$$

[∵ cylinder is open at the top, therefore
S = curved surface area of cylinder
+ area of base]

$$\Rightarrow \quad S = 2\pi r\left(\frac{V}{\pi r^2}\right) + \pi r^2$$

$$\left[\text{putting } h = \frac{V}{\pi r^2}, \text{ from Eq. (i)}\right]$$

$$\Rightarrow \quad S = \frac{2V}{r} + \pi r^2 \qquad \textbf{(1)}$$

On differentiating w.r.t. r, we get

$$\frac{dS}{dr} = -\frac{2V}{r^2} + 2\pi r$$

For maxima and minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow -\frac{2V}{r^2} + 2\pi r = 0$$

$$\Rightarrow \quad V = \pi r^3$$

$$\Rightarrow \quad \pi r^2 h = \pi r^3 \qquad [\because V = \pi r^2 h, \text{ given}]$$

$$\Rightarrow \quad h = r \qquad \textbf{(1)}$$

Also, $$\frac{d^2S}{dr^2} = \frac{d}{dr}\left(\frac{dS}{dr}\right) = \frac{d}{dr}\left(\frac{-2V}{r^2} + 2\pi r\right)$$

$$\Rightarrow \quad \frac{d^2S}{dr^2} = \frac{4V}{r^3} + 2\pi \qquad \textbf{(1)}$$

On putting $r = h$, we get

$$\left[\frac{d^2S}{dr^2}\right]_{r=h} = \frac{4V}{h^3} + 2\pi > 0 \text{ as } h > 0$$ **(1)**

Then, $\frac{d^2S}{dr^2} > 0 \Rightarrow S$ is minimum.

Hence, S is minimum, when $h = r$, i.e. when height of cylinder is equal to radius of the base.

(1)

14. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

All India 2013

Let $ABCD$ be a rectangle having area A inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(i)

Let the coordinates of A be (α, β).

Then, coordinates of $B = (\alpha_1 - \beta)$

$C = (\alpha_1 - \beta)$

$D = (\alpha_1 - \beta)$ **(1)**

Area, A = Length× Breadth $= 2\alpha \times 2\beta$

$\Rightarrow \quad A = 4\alpha\beta$

$\Rightarrow \quad A = 4\alpha \cdot \sqrt{b^2\left(1 - \frac{\alpha^2}{a^2}\right)}$ **(1)**

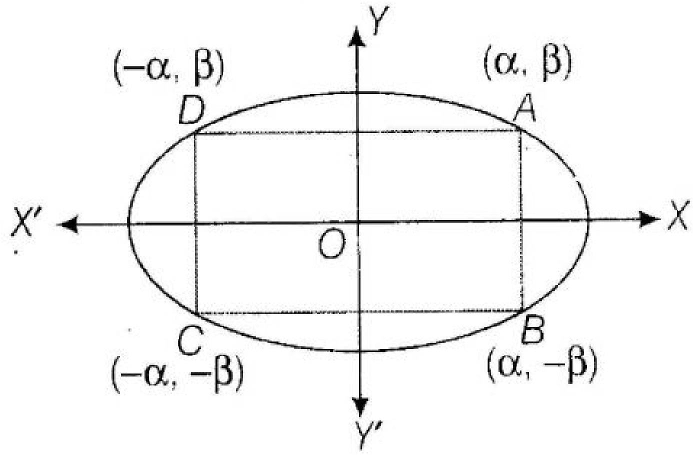
$$\left[\begin{array}{r} \because (\alpha, \beta) \text{ lies on ellipse} \\ \therefore \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 1, \text{ i.e. } \beta = \sqrt{b^2\left(\frac{1-\alpha^2}{a^2}\right)} \end{array}\right]$$

$\Rightarrow \quad A^2 = 16\alpha^2\left\{b^2\left(\frac{1-\alpha^2}{a^2}\right)\right\}$

[on squaring both sides]

$16b^2$

$$\Rightarrow \qquad A^2 = \frac{16b^2}{a^2}(a^2\alpha^2 - \alpha^4)$$



On differentiating w.r.t. α, we get

$$\frac{d(A^2)}{d\alpha} = \frac{16b^2}{a^2}(2a^2\alpha - 4\alpha^3)$$

For maximum or minimum value, put

$$\frac{dA^2}{d\alpha} = 0$$

$$\Rightarrow \qquad 2a^2\alpha - 4\alpha^3 = 0$$

$$\Rightarrow \qquad 2\alpha(a^2 - 2\alpha^2) = 0$$

$$\Rightarrow \qquad \alpha = 0, \alpha = \frac{a}{\sqrt{2}} \qquad \textbf{(1)}$$

Again, $\dfrac{d^2(A^2)}{d\alpha^2} = \dfrac{16b^2}{a^2}(2a^2 - 12\alpha^2)$

At $\qquad \alpha = \dfrac{a^2}{\sqrt{2}}, \left(\dfrac{d^2A^2}{d\alpha^2}\right)_{\alpha = \frac{a}{\sqrt{2}}}$

$$= \frac{16b^2}{a^2}\left(2a^2 - 12 \times \frac{a^2}{2}\right) = 0 \qquad \textbf{(1)}$$

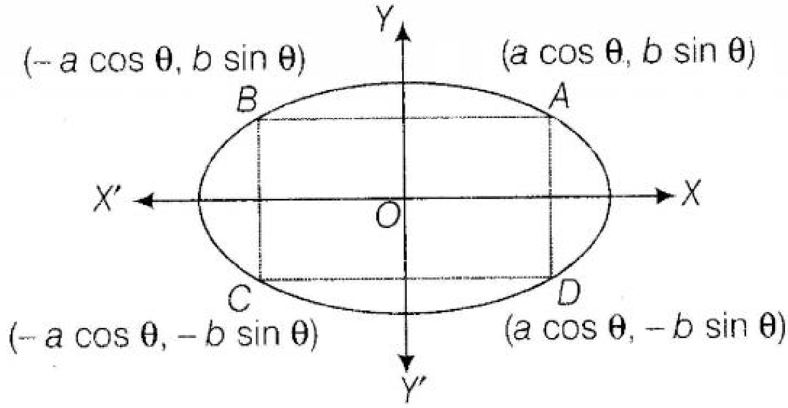
$\Rightarrow$ For $\alpha = \dfrac{a}{\sqrt{2}}, A^2$ i.e. A is maximum.

Then, from Eq. (i), we get $\beta = \dfrac{b}{\sqrt{2}}$

$\therefore$ Greatest area $= 4\alpha\beta = 4.\dfrac{\alpha}{\sqrt{2}}.\dfrac{b}{\sqrt{2}} = 2ab$ **(1)**

Alternate Method

Let $A(a\cos\theta, b\sin\theta)$ be the parametric coordinates of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where θ is the eccentric angle. **(1)**



(1)

Here, length of $AB = a\cos\theta + a\cos\theta$

$= 2a\cos\theta$

and length of $AD = b\sin\theta + b\sin\theta$

$= 2b\sin\theta$ **(1)**

Now, area of rectangle $ABCD$

$= (2a\cos\theta)(2b\sin\theta)$

$= 2ab(2\sin\theta\cdot\cos\theta)$

$= 2ab\sin 2\theta$ **(1½)**

Here, area of rectangle $ABCD$ is greatest, when $\sin 2\theta$ is greatest.

i.e. $\sin 2\theta = 1 = \sin 90°$

$\Rightarrow$ $2\theta = 90° \Rightarrow \theta = 45°$ **(1½)**

Hence, area of greatest rectangle is equal to $2ab$, when eccentric angle of an ellipse is $45°$.

15. Show that the height of a closed right circular cylinder of given surface and maximum volume is equal to diameter of base. **Delhi 2012**

Here, use the relations, total surface area of cylinder, $S = 2\pi r^2 + 2\pi rh$ and volume of cylinder $V = \pi r^2 h$. To make a relation between V and S and differentiate with respect to r. Then, put $\frac{dV}{dr}=0$. And determine r and then check $\frac{d^2V}{dr^2}$, if it is negative, then maxima and if it is positive, then minima.

Let S be the surface area, V be the volume, h be the height and r be the radius of base of the right circular cylinder. We know that,

Surface area of right circular cylinder,

$$S = 2\pi r^2 + 2\pi rh \quad \text{...(i)}$$

$$\Rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r} \quad \text{...(ii)} \ \textbf{(1)}$$

Also, volume of right circular cylinder is given by

$$V = \pi r^2 h$$

$$\Rightarrow \quad V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r} \right) \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad V = \frac{rS - 2\pi r^3}{2} \quad \textbf{(1)}$$

On differentiating w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S - 6\pi r^2}{2} \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow \quad \frac{S - 6\pi r^2}{2} = 0 \Rightarrow S = 6\pi r^2 \quad \textbf{(1)}$$

From Eq. (ii), we get

$$h = \frac{6\pi r^2 - \pi r^2}{2\pi r} \quad \Rightarrow \quad h = 2r$$

$\therefore$ Height = Diameter of the base **(1)**

Also, $\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(\frac{S - 6\pi r^2}{2}\right)$

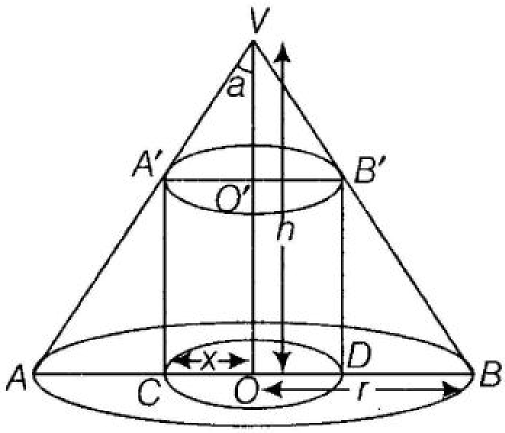
$$= -6\pi r < 0$$

$\Rightarrow V$ is maximum. **(1)**

Hence, V is maximum at $h = 2r$.

16. Prove that radius of right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. All India 2012

Let VAB be the cone of base radius r and height h. And radius of base of the inscribed cylinder be x.



(1)

Now, we observe that

$$\Delta VOB \sim \Delta B'DB \Rightarrow \frac{VO}{B'D} = \frac{OB}{DB}$$

$$\Rightarrow \quad \frac{h}{B'D} = \frac{r}{r-x} \quad \Rightarrow B'D = \frac{h(r-x)}{r} \quad \textbf{(1)}$$

Let C be the curved surface area of cylinder. Then,

$$C = 2\pi (OC)(B'D)$$

$$\Rightarrow \quad C = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2) \quad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dC}{dx} = \frac{2\pi h}{r}(r - 2x)$$

For maxima and minima, put $\frac{dC}{dx} = 0$

$$\Rightarrow \frac{2\pi h}{r}(r-2x) = 0 \quad \Rightarrow \quad r - 2x = 0$$

$$\Rightarrow \quad r = 2x \Rightarrow \quad x = \frac{r}{2} \quad \textbf{(1)}$$

Hence, radius of cylinder is half of that of cone.

Also, $\frac{d^2C}{dx^2} = \frac{d}{dx}\left[\frac{2\pi h(r-2x)}{r}\right]$

$$= \frac{2\pi h}{r}(-2) = \frac{-4\pi h}{r} < 0 \text{ as } h, r > 0 \quad \textbf{(1)}$$

$\Rightarrow C$ is maximum or greatest.

Hence, C is greatest at $x = \frac{r}{2}$. **(1)**

Hence proved.

17. An open box with a square base is to be made out of a given quantity of cardboard of area C^2 sq units. Show that the maximum volume of box is $\frac{C^3}{6\sqrt{3}}$ cu units. All India 2012

Let the dimensions of the box be x and y. Also, let V denotes its volume and S denotes its total surface area.

Now, $S = x^2 + 4xy$ [$\because S$ = area of square base + area of the four walls]

Given, $x^2 + 4xy = C^2$

$$\Rightarrow \quad y = \frac{C^2 - x^2}{4x} \qquad \text{...(i)} \ \textbf{(1)}$$

Also, volume of the box is given by

$$V = x^2 y \Rightarrow V = x^2 \left(\frac{C^2 - x^2}{4x} \right) \text{[from Eq. (i)]}$$

$$\Rightarrow \quad V = \frac{xC^2 - x^3}{4} \qquad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dV}{dx} = \frac{C^2 - 3x^2}{4} \qquad \textbf{(1)}$$

For maxima and minima, put $dV/dx = 0$

$$\Rightarrow \quad \frac{C^2 - 3x^2}{4} = 0 \quad \Rightarrow \quad C^2 = 3x^2$$

$$\therefore \quad x = C/\sqrt{3} \qquad \textbf{(1)}$$

Also, $$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right) = \frac{d}{dx}\left(\frac{C^2 - 3x^2}{4}\right)$$

$$= \frac{-6x}{4} = \frac{-3x}{2} < 0$$

$$\because \quad \frac{d^2V}{dx^2} < 0 \Rightarrow V \text{ is maximum.} \qquad \textbf{(1)}$$

Now, maximum volume,

$$V = \frac{xC^2 - x^3}{4} = \frac{1}{4}[xC^2 - x^3]$$

$$= \frac{1}{4}\left[\frac{C}{\sqrt{3}} \cdot C^2 - \left(\frac{C}{\sqrt{3}}\right)^3\right] \quad \left[\text{put } x = \frac{C}{\sqrt{3}}\right]$$

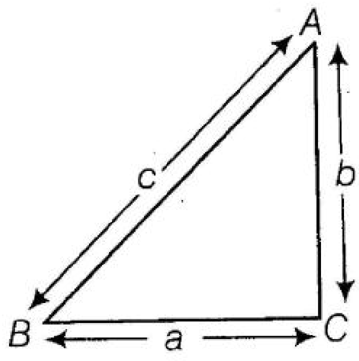
$$= \frac{1}{4}\left[\frac{C^3}{\sqrt{3}} - \frac{C^3}{3\sqrt{3}}\right] = \frac{1}{4}\left[\frac{3C^3 - C^3}{3\sqrt{3}}\right]$$

$$= \frac{1}{4} \times \frac{2C^3}{3\sqrt{3}} = \frac{C^3}{6\sqrt{3}}$$

Hence, maximum volume of box is $\frac{C^3}{6\sqrt{3}}$ cu units. **(1)**

18. Prove that the area of a right-angled triangle of given hypotenuse is maximum, when the triangle is isosceles. **Delhi 2012C**

Let a and b be the sides of right angled triangle.



(1/2)

From ΔABC, we have

$$c^2 = a^2 + b^2$$

Area of ΔABC $(A) = \frac{1}{2} a \cdot b = \frac{1}{2} a\sqrt{c^2 - a^2}$

$[\because b = \sqrt{c^2 - a^2}]$ **(1)**

On differentiating w.r.t. a, we get

$$\frac{dA}{da} = \frac{1}{2} \cdot 1 \cdot \sqrt{c^2 - a^2} + \frac{1}{2} \cdot a \cdot \frac{1}{2} \cdot \frac{(-2a)}{\sqrt{c^2 - a^2}}$$

$$= \frac{1}{2}\left[\sqrt{c^2 - a^2} - \frac{a^2}{\sqrt{c^2 - a^2}}\right]$$ **(1)**

For maxima and minima, $\frac{dA}{da} = 0$

$$\Rightarrow \frac{1}{2}\left[\sqrt{c^2 - a^2} - \frac{a^2}{\sqrt{c^2 - a^2}}\right] = 0$$

$$\Rightarrow c^2 - a^2 - a^2 = 0 \Rightarrow c^2 = 2a^2$$

$$\Rightarrow a = \frac{c}{\sqrt{2}}$$

Now, $$\frac{d^2A}{da^2} = \frac{1}{2}\left[\frac{-a}{\sqrt{c^2 - a^2}} - \frac{a^3}{(c^2 - a^2)^{3/2}}\right]$$

$$= -\frac{1}{2} a\left[\frac{c^2 - a^2 + a^2}{(c^2 - a^2)^{3/2}}\right]$$ **(1)**

$$= -\frac{1}{2} \frac{c^2 a}{(c^2 - a^2)^{3/2}} < 0$$ **(1½)**

$\therefore$ Area of ΔABC is maximum and

$b = \sqrt{c^2 - a^2} = \sqrt{2a^2 - a^2} = a$ **(1)**

Hence, triangle is isosceles. **Hence proved.**

19. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. **HOTS; Delhi 2011**

Using the result, volume of cone, $V = \frac{1}{3}\pi r^2 h$ and curved surface area, $S = \pi r l$. Make the relation between V and S, differentiate it and simplify to get the result.

Let C denotes the curved surface area, r be the radius of base, h be the height and V be the volume of right circular cone.

To show, $\quad h = \sqrt{2}r$

We know that, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2} \quad \ldots\text{(i)} \; \mathbf{(1)}$$

Also, the curved surface area of cone is given by $C = \pi r l$, where $l = \sqrt{r^2 + h^2}$ is the slant height of cone.

$$\therefore \quad C = \pi r \sqrt{r^2 + h^2}$$

On squaring both sides, we get

$$C^2 = \pi^2 r^2 (r^2 + h^2) \Rightarrow C^2 = \pi^2 r^4 + \pi^2 r^2 h^2$$

Let $\quad C^2 = Z$

Then, $\quad Z = \pi^2 r^4 + \pi^2 r^2 h^2 \quad \ldots\text{(ii)}$

$$\Rightarrow \quad Z = \pi^2 r^4 + \pi^2 r^2 \left(\frac{3V}{\pi r^2}\right)^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad Z = \pi^2 r^4 + \pi^2 r^2 \times \frac{9V^2}{\pi^2 r^4} \quad \mathbf{(1½)}$$

$$\Rightarrow \quad Z = \pi^2 r^4 + \frac{9V^2}{r^2}$$

On differentiating both sides w.r.t. r, we get

$$\frac{dZ}{dr} = 4\pi^2 r^3 - \frac{18V^2}{r^3} \quad (½)$$

For maxima and minima, put $\frac{dZ}{dr} = 0$

$$\Rightarrow \quad 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0 \Rightarrow 4\pi^2 r^3 = \frac{18V^2}{r^3}$$

$$\Rightarrow \quad 4\pi^2 r^6 = 18\left(\frac{1}{3}\pi r^2 h\right) \quad \left[\because V = \frac{1}{3}\pi r^2 h\right]$$

$$\Rightarrow \quad 4\pi^2 r^6 = 18 \times \frac{1}{9}\pi^2 r^4 h^2$$

$$\Rightarrow \quad 4\pi^2 r^6 = 2\pi^2 r^4 h^2 \Rightarrow 2r^2 = h^2 \Rightarrow h = \sqrt{2}r$$

Hence, height $= \sqrt{2}$ [radius of base] **(1½)**

Also, $\dfrac{d^2 Z}{dr^2} = \dfrac{d}{dr}\left(\dfrac{dz}{dr}\right)$

$$= \frac{d}{dr}\left(4\pi^2 r^3 - \frac{18V^2}{r^3}\right) = 12\pi^2 r^2 + \frac{54V^2}{r^4}$$

$$\therefore \frac{d^2 Z}{dr^2} = 12\pi^2 r^2 + \frac{54V^2}{r^4} > 0$$

$\Rightarrow Z$ is minimum $\Rightarrow C$ is minimum.

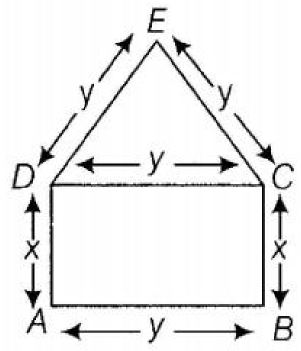
Hence, curved surface area is least, when $h = \sqrt{2}r$. **(1½)**

NOTE If C is maximum/minimum, then C^2 is also maximum/minimum.

20. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, then find the dimensions of the rectangle that will produce the largest area of the window.

Delhi 2011

Let $ABCD$ be the rectangle which is surmounted by an equilateral ΔEDC.



Now, given that

Perimeter of window = 12 m

$$\Rightarrow \quad 2x + 3y + y = 12$$

$$\therefore \quad x = 6 - 2y \quad \ldots(i) \quad \textbf{(1)}$$

Let A denote the combined area of the window. Then,

$$A = xy + \frac{\sqrt{3}}{4}y^2$$

$$\left[\because \text{combined area} = \text{area of rectangle} + \text{area of equilateral triangle}\right]$$

$$\Rightarrow \quad A = y(6-2y) + \frac{\sqrt{3}}{4}y^2 \quad \textbf{(1)}$$

$[\because x = 6 - 2y$ from Eq. (i)$]$

$$\Rightarrow \quad A = 6y - 2y^2 + \frac{\sqrt{3}}{4}y^2$$

On differentiating w.r.t. y, we get

$$\frac{dA}{dy} = 6 - 4y + \frac{2\sqrt{3}}{4}y \quad \textbf{(1)}$$

Now, for maxima and minima, put $\frac{dA}{dy} = 0$

$$\Rightarrow \quad 6 - 4y + \frac{2\sqrt{3}}{4}y = 0$$

$$\Rightarrow \quad y\left(\frac{\sqrt{3}}{2} - 4\right) = -6 \quad \Rightarrow \quad y = \frac{12}{8-\sqrt{3}}$$

Now, $\frac{d^2A}{dy^2} = \frac{d}{dy}\left(\frac{dA}{dy}\right) = \frac{d}{dy}\left(6 - 4y + \frac{2\sqrt{3}}{4}y\right)$

$$= -4 + \frac{2\sqrt{3}}{4} = \frac{-8+\sqrt{3}}{2} < 0$$

$\therefore A$ is maximum. **(1)**

Now, on putting $y = \frac{12}{8-\sqrt{3}}$ in Eq. (i), we get

$$x = 6 - 2\left(\frac{12}{8-\sqrt{3}}\right) \Rightarrow x = \frac{48 - 6\sqrt{3} - 24}{8-\sqrt{3}}$$

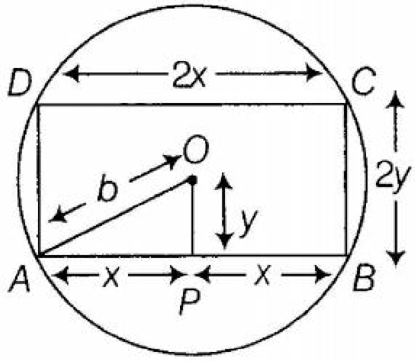
$$\Rightarrow \quad x = \frac{24 - 6\sqrt{3}}{8-\sqrt{3}} \quad \textbf{(1)}$$

Hence, the area of the window is largest and the dimensions of the window are

$$x = \frac{24-6\sqrt{3}}{8-\sqrt{3}} \quad \text{and} \quad y = \frac{12}{8-\sqrt{3}}. \quad \textbf{(1)}$$

21. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area. **All India 2011**

Let $ABCD$ be the rectangle which is inscribed in the fixed circle which has centre O and radius b. Let $AB = 2x$ and $BC = 2y$.



In right angled ΔOPA, by Pythagoras theorem, we have

$$OP^2 + AP^2 = OA^2$$

$$\Rightarrow \quad x^2 + y^2 = b^2 \Rightarrow y^2 = b^2 - x^2$$

$$\Rightarrow \quad y = \sqrt{b^2 - x^2} \quad \text{...(i)} \ \textbf{(1)}$$

Let A be the area of rectangle.

$$\therefore \quad A = (2x)(2y)$$

[$\because$ area of rectangle = Length × Breadth]

$$\Rightarrow \quad A = 4xy$$

$$\Rightarrow \quad A = 4x\sqrt{b^2 - x^2} \qquad [\because\ y = \sqrt{b^2 - x^2}]$$

On differentiating w.r.t. x, we get

$$\frac{dA}{dx} = 4x \cdot \frac{d}{dx}\sqrt{b^2 - x^2} + \sqrt{b^2 - x^2} \cdot \frac{d}{dx}(4x)$$

$$\Rightarrow \frac{dA}{dx} = 4x \cdot \frac{-2x}{2\sqrt{b^2 - x^2}} + \sqrt{b^2 - x^2} \cdot 4$$

$$= 4\left[\frac{b^2 - x^2 - x^2}{\sqrt{b^2 - x^2}}\right]$$

$$\Rightarrow \frac{dA}{dx} = 4\left(\frac{b^2 - 2x^2}{\sqrt{b^2 - x^2}}\right) \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dA}{dx} = 0$

$$\therefore \quad 4\left(\frac{b^2 - 2x^2}{\sqrt{b^2 - x^2}}\right) = 0$$

$$\Rightarrow \quad b^2 - 2x^2 = 0 \Rightarrow 2x^2 = b^2$$

$$\Rightarrow \quad x = \frac{b}{\sqrt{2}} \quad \textbf{(1)}$$

[$\because$ x cannot be negative]

Also, $$\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{dA}{dx}\right) = \frac{d}{dx}\left[\frac{4(b^2 - 2x^2)}{\sqrt{b^2 - x^2}}\right]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{d}{dx}[4(b^2 - 2x^2)(b^2 - x^2)^{-1/2}]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = 4[-4x(b^2 - x^2)^{-1/2} + (b^2 - 2x^2)\left(-\frac{1}{2}\right)(b^2 - x^2)^{-3/2}(-2x)]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = 4\left[\frac{-4x}{\sqrt{b^2 - x^2}} + \frac{x(b^2 - 2x^2)}{(b^2 - x^2)^{3/2}}\right] \quad \textbf{(1)}$$

On putting $x = \frac{b}{\sqrt{2}}$, we get

$$\frac{d^2A}{dx^2} = 4\left[\frac{\frac{-4b}{\sqrt{2}}}{\sqrt{b^2 - \frac{b^2}{2}}} + \frac{\frac{b}{\sqrt{2}}\left(b^2 - 2 \times \frac{b^2}{2}\right)}{\left(b^2 - \frac{b^2}{2}\right)^{3/2}}\right]$$

$$= 4\left[\frac{\frac{-4b}{\sqrt{2}}}{\sqrt{\frac{b^2}{2}}} + 0\right]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = -16 < 0$$

$\because \frac{d^2A}{dx^2} < 0$. So, A is maximum at $x = \frac{b}{\sqrt{2}}$. **(1)**

$\frac{d^2A}{dx^2} < 0$, A is maximum at $x = \frac{b}{\sqrt{2}}$ (1)

Now, putting $x = \frac{b}{\sqrt{2}}$ in Eq. (i), we get

$$y = \sqrt{b^2 - \frac{b^2}{2}} = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}}$$

$$\therefore \qquad x = y = \frac{b}{\sqrt{2}} \Rightarrow 2x = 2y = \sqrt{2}b$$

Hence, area of rectangle is maximum, when $2x = 2y$, i.e. when rectangle is a square. **(1)**

22. Show that of all the rectangles with a given perimeter, the square has the largest area.

Delhi 2011

Using the formula perimeter of rectangle, $P = 2(x + y)$ and area of rectangle, $A = xy$. Making a relation between A and P, differentiate A with respect to x and simplify it.

Let x and y be the lengths of two sides of a rectangle. Again, let P denotes its perimeter and A be the area of rectangle. Given,

$$P = 2(x + y) \quad \textbf{(1)}$$

[$\because$ perimeter of rectangle $= 2(l + b)$]

$$\Rightarrow \quad P = 2x + 2y$$

$$\Rightarrow \quad y = \frac{P - 2x}{2} \quad \text{...(i)} \ \textbf{(1)}$$

Also, we know that area of rectangle is given by

$$A = xy$$

$$\Rightarrow \quad A = x\left(\frac{P - 2x}{2}\right) \quad \text{[by using Eq. (i)]}$$

$$\Rightarrow \quad A = \frac{Px - 2x^2}{2} \quad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dA}{dx} = \frac{P - 4x}{2} \quad \textbf{(1)}$$

Now, for maxima and minima, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{P - 4x}{2} = 0 \Rightarrow P = 4x \quad \textbf{(1)}$$

$$\Rightarrow \quad 2x + 2y = 4x \quad [\because P = 2x + 2y]$$

$$\Rightarrow \quad x = y$$

So, the rectangle is a square.

Also, $\frac{d^2A}{dx^2} = \frac{d}{dx}\left(\frac{P - 4x}{2}\right) = -\frac{4}{2} = -2 < 0$

$\Rightarrow$ A is maximum.

Hence, area is maximum, when rectangle is a square. **(1)**

23. Show that of all the rectangles of given area, the square has the smallest perimeter.

Delhi 2011

Let x and y be the lengths of sides of a rectangle. A denotes its area and P be the perimeter.

Now, $A = xy$

[$\because$ area of rectangle $= l \times b$]

$\Rightarrow \quad y = \frac{A}{x}$ ...(i) **(1)**

And, $\quad P = 2(x + y)$

[$\because$ perimeter of rectangle $= 2(l + b)$]

$\Rightarrow \quad P = 2\left(x + \frac{A}{x}\right) \quad \left[\because y = \frac{A}{x} \text{ by Eq.(i)}\right]$ **(1)**

On differentiating w.r.t. x, we get

$$\frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right)$$ **(1)**

For maxima and minima, put $\frac{dP}{dx} = 0$

$\Rightarrow \quad 2\left(1 - \frac{A}{x^2}\right) = 0 \Rightarrow 1 = \frac{A}{x^2}$ **(1)**

$\Rightarrow \quad A = x^2$

$\Rightarrow \quad xy = x^2$ [$\because A = xy$]

$\Rightarrow \quad x = y$ **(1)**

Also, $\frac{d^2P}{dx^2} = \frac{d}{dx}\left[2\left(1 - \frac{A}{x^2}\right)\right] = 2\left(\frac{2A}{x^3}\right) = \frac{4A}{x^3} > 0$

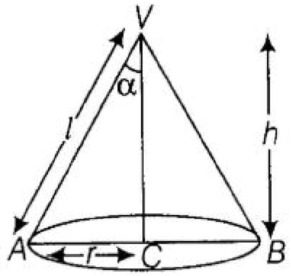
Here, x and A being the side and area of rectangle can never be negative. So, P is minimum.

Hence, perimeter of rectangle is minimum, when rectangle is a square. **(1)**

24. Show that the semi-vertical angle of a right circular cone of maximum volume and given slant height is $\tan^{-1}\sqrt{2}$.

All India 2011, 2008; Delhi 2008C

Let h be the height, l be the slant height, r be the radius of base of the right circular cone and α be the semi-vertical angle of the cone.



In ΔVAC, by Pythagoras theorem, we have

$$l^2 = r^2 + h^2 \Rightarrow r^2 = l^2 - h^2 \quad \text{...(i)} \quad \textbf{(1)}$$

Let V be the volume of cone which is given by

$$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{\pi}{3}(l^2 - h^2)\cdot h$$

$$\Rightarrow \quad V = \frac{\pi}{3}(l^2 h - h^3) \quad \textbf{(1)}$$

On differentiating w.r.t. h, we get

$$\frac{dV}{dh} = \frac{\pi}{3}(l^2 - 3h^2) \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dV}{dh} = 0$

$$\Rightarrow \quad \frac{\pi}{3}(l^2 - 3h^2) = 0 \Rightarrow l^2 = 3h^2$$

$$\Rightarrow \quad r^2 + h^2 = 3h^2 \quad [\because l^2 = r^2 + h^2]$$

$$\Rightarrow \quad 2h^2 = r^2 \Rightarrow r = \sqrt{2}h \quad \text{...(ii)} \quad \textbf{(1)}$$

Now, in right angled ΔCVA, we have

$$\tan\alpha = \frac{AC}{VC}$$

$$\Rightarrow \quad \tan\alpha = \frac{r}{h} \quad [\because AC = r \text{ and } VC = h]$$

$$\Rightarrow \quad \tan\alpha = \frac{\sqrt{2}h}{h} \quad [\because r = \sqrt{2}h, \text{ by Eq. (ii)}]$$

$$\Rightarrow \qquad \tan\alpha = \sqrt{2} \qquad \textbf{(1)}$$

$$\Rightarrow \qquad \alpha = \tan^{-1}\sqrt{2}$$

Also, $$\frac{d^2V}{dh^2} = \frac{d}{dh}\left[\frac{\pi}{3}(l^2 - 3h^2)\right]$$

$$= \frac{\pi}{3}(-6h) = -2\pi h < 0 \text{ as } h > 0.$$

$\therefore V$ is maximum.

Hence, the volume is maximum, when

$$\alpha = \tan^{-1}\sqrt{2} \qquad \textbf{(1)}$$

25. Find the point on the curve $y^2 = 2x$ which is at a minimum distance from the point $(1, 4)$.

HOTS; All India 2011, 2009C

Firstly, consider any point on the curve, use the formula of distance between two points. Then, square both sides and eliminate one variable with the help of given equation. Further, differentiate and solve it to get the result.

The given equation of curve is $y^2 = 2x$ and the given point is Q (1, 4).

Let $P(x, y)$ be the point, which is at a minimum distance from point Q (1, 4). **(1)**

Now, distance between points P and Q is given by

$$PQ = \sqrt{(1-x)^2 + (4-y)^2}$$

[using distance formula

$$S = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}]$$

$$\Rightarrow \quad PQ = \sqrt{1 + x^2 - 2x + 16 + y^2 - 8y}$$

$$= \sqrt{x^2 + y^2 - 2x - 8y + 17}$$

On squaring both sides, we get

$$PQ^2 = x^2 + y^2 - 2x - 8y + 17$$

$$\Rightarrow \quad PQ^2 = \left(\frac{y^2}{2}\right)^2 + y^2 - 2\left(\frac{y^2}{2}\right) - 8y + 17$$

$$\left[\because\ y^2 = 2x \text{ is given } \Rightarrow x = \frac{y^2}{2}\right]$$

$$\therefore \quad PQ^2 = \frac{y^4}{4} + y^2 - y^2 - 8y + 17$$

$$\Rightarrow \quad PQ^2 = \frac{y^4}{4} - 8y + 17 \qquad \textbf{(1)}$$

Let $PQ^2 = Z$

Then, $Z = \frac{y^4}{4} - 8y + 17$

On differentiating w.r.t. y, we get

$$\frac{dZ}{dy} = \frac{4y^3}{4} - 8 = y^3 - 8 \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dZ}{dy} = 0$

$\Rightarrow \quad y^3 - 8 = 0$

$\Rightarrow \quad y^3 = 8$

$\Rightarrow \quad y = 2$ **(1)**

Also, $\frac{d^2Z}{dy^2} = \frac{d}{dy}(y^3 - 8) = 3y^2$

On putting $y = 2$, we get

$$\left[\frac{d^2Z}{dy^2}\right]_{y=2} = 3(2)^2 = 12 > 0$$

$\because \quad \frac{d^2Z}{dy^2} > 0$

$\therefore$ Z is minimum and therefore PQ is also minimum as $Z = PQ^2$. **(1)**

On putting $y = 2$ in the given equation, i.e. $y^2 = 2x$, we get

$$(2)^2 = 2x \Rightarrow 4 = 2x \Rightarrow x = 2$$

Hence, the point which is at a minimum distance from point (1, 4) is P (2, 2). **(1)**

26. A wire of length 28 m is to be cut into two pieces. One of the two pieces is to be made into a square and the other into a circle. What should be the lengths of two pieces, so that the combined area of circle and square is minimum? **All India 2010**

Firstly, find length of circular part and its circumference and calculate the length of square part and its perimeter. Add these two terms and equate it to 28 m and apply second derivative test to get desired result.

Let x m be the side of the square and r be the radius of circular part. Then,

Length of square part = Perimeter of square

$$= 4 \times \text{Side} = 4x$$

and length of circular part

$$= \text{Circumference of circle} = 2\pi r$$

Given, length of wire = 28 $\Rightarrow 4x + 2\pi r = 28$

$$\Rightarrow \quad 2x + \pi r = 14$$

$$\therefore \quad x = \frac{14 - \pi r}{2} \quad \text{...(i)} \; \textbf{(1)}$$

Let A denotes the combined area of circle and square.

Then, $\quad A = \pi r^2 + x^2$

$$\Rightarrow \quad A = \pi r^2 + \left(\frac{14 - \pi r}{2}\right)^2$$

$$\left[\because x = \frac{14 - \pi r}{2}, \text{ from Eq. (i)}\right]$$

On differentiating w.r.t. r, we get

$$\frac{dA}{dr} = 2\pi r + 2\left(\frac{14 - \pi r}{2}\right)\left(-\frac{\pi}{2}\right)$$

$$= 2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right) \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dA}{dr} = 0$

$$\Rightarrow \quad 2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right) = 0$$

$$\Rightarrow \quad 2\pi r = \frac{14\pi - \pi^2 r}{2}$$

$\Rightarrow \quad r = \frac{14}{\pi + 4}$ **(1)**

Also,
$$\frac{d^2A}{dr^2} = \frac{d}{dr}\left(\frac{dA}{dr}\right)$$
$$= \frac{d}{dr}\left[2\pi r - \left(\frac{14\pi - \pi^2 r}{2}\right)\right]$$

$$\Rightarrow \quad \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$

Thus, $\frac{d^2A}{dr^2} > 0 \Rightarrow A$ is minimum. **(1)**

Now, on putting $r = \frac{14}{\pi + 4}$ in Eq. (i), we get

$$x = \frac{14 - \pi\left(\frac{14}{\pi + 4}\right)}{2} = \frac{14\pi + 56 - 14\pi}{2\,(\pi + 4)} = \frac{28}{\pi + 4}$$

(1/2)

$\therefore \quad x = \frac{28}{\pi + 4}$ and $r = \frac{14}{\pi + 4}$ **(1/2)**

Now, length of circular part

$$= 2\pi r = 2\pi \times \frac{14}{\pi + 4} = \frac{28\pi}{\pi + 4}$$

and length of square part $= 4x = 4 \times \frac{28}{\pi + 4}$

$$= \frac{112}{\pi + 4}$$ **(1)**

which are the required length of two pieces.

27. An open tank with a square base and vertical sides is to be constructed from a metal sheet, so as to hold a given quantity of water. Show that the total surface area is least when depth of the tank is half its width. **All India 2010C**

Let the length, breadth and depth of the open tank be x, x and y, respectively. Length and breadth are same because given tank has a square base. Again, let V denotes its volume and S denotes its surface area. Now, given that

$$V = x^2 y \qquad \text{...(i)} \ \mathbf{(1)}$$

Also, we know that the total surface area of the open tank is given by

$$S = x^2 + 4xy \qquad \text{...(ii)}$$

[$\because S$ = Area of square base + Area of the four walls]

On putting $y = \frac{V}{x^2}$ from Eq. (i) in Eq. (ii), we get

$$S = x^2 + 4x \cdot \frac{V}{x^2}$$

$$\Rightarrow \quad S = x^2 + \frac{4V}{x} \qquad \mathbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dS}{dx} = 2x - \frac{4V}{x^2}$$

For maxima and minima, put $\frac{dS}{dx} = 0$

$$\Rightarrow \quad 2x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \quad 4V = 2x^3 \qquad \mathbf{(1)}$$

$$\Rightarrow \quad 4x^2 y = 2x^3 \ [\because V = x^2 y, \text{ from Eq. (i)}]$$

$$\Rightarrow \quad 2y = x \Rightarrow y = \frac{x}{2}$$

So, depth of tank is half of its width. **(1)**

Also, $\frac{d^2S}{dx^2} = \frac{d}{dx}\left(\frac{dS}{dx}\right) = \frac{d}{dx}\left(2x - \frac{4V}{x^2}\right)$

$$= 2 + \frac{8V}{x^3}$$

$$= 2 + \frac{8 \cdot x^2 y}{x^3} \qquad \text{[from Eq. (i)]}$$

$$= 2 + \frac{8y}{x} > 0 \text{ as } x > 0 \text{ and } y > 0 \qquad \mathbf{(1)}$$

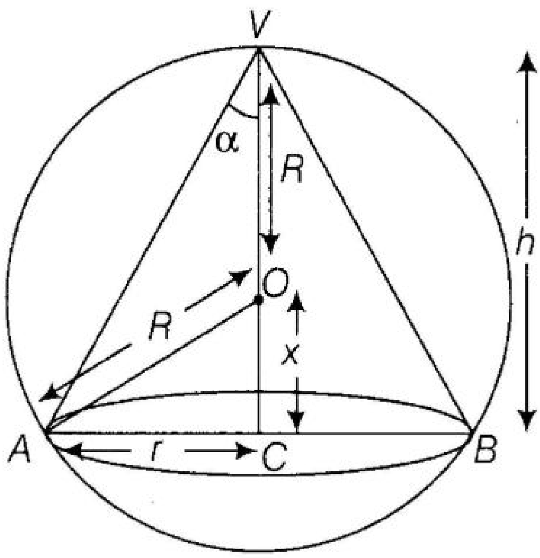
Thus, $\frac{d^2S}{dx^2} > 0$,

$\Rightarrow$ S is minimum.

Hence, total surface area of the tank is less when depth is half of its width. (1)

28. Show that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere. **Delhi 2010C**

Let R be the radius of sphere, r be the radius of base of cone and h be the height of cone.



Then, from the figure,

$$h = R + x \quad \text{...(i)}\textbf{(1/2)}$$

Let V denotes the volume of cone. Now, in right angled ΔOCA ,we get

$$OA^2 = OC^2 + AC^2$$

[by Pythagoras theorem]

$$\Rightarrow \quad R^2 = x^2 + r^2$$

$$\Rightarrow \quad r^2 = R^2 - x^2 \quad \text{...(ii)}\ \textbf{(1)}$$

Also, we know that, volume of cone is given by

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad V = \frac{1}{3}\pi (R^2 - x^2)(R + x)$$

$$\left[\begin{array}{ll} \because & h = R + x, \text{ from Eq.(i)} \\ \text{and} & r^2 = R^2 - x^2, \text{ from Eq.(ii)} \end{array}\right]$$

$$\Rightarrow \quad V = \frac{\pi}{3}(R^3 + R^2x - x^2R - x^3)$$

On differentiating w.r.t. x, we get

$$\frac{dV}{dx} = \frac{\pi}{3}(R^2 - 2xR - 3x^2)$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}[R^2 - 3xR + xR - 3x^2]$$

$$= \frac{\pi}{3}[R(R - 3x) + x(R - 3x)]$$

$$\Rightarrow \quad \frac{dV}{dx} = \frac{\pi}{3}(R + x)(R - 3x) \qquad \textbf{(1½)}$$

For maxima and minima, put $\frac{dV}{dx} = 0$

$$\Rightarrow \quad \frac{\pi}{3}(R + x)(R - 3x) = 0$$

$\Rightarrow$ Either $R + x = 0$ or $R - 3x = 0$

Now, $R + x = h$ which is height of cone. As h can never be zero. So, $R + x = 0$ is rejected.

$$\therefore \quad R - 3x = 0$$

$$\Rightarrow \quad 3x = R$$

$$\Rightarrow \quad x = \frac{R}{3} \qquad \textbf{(1)}$$

Also,

$$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right)$$

$$= \frac{d}{dx}\left[\frac{\pi}{3}(R^2 - 2xR - 3x^2)\right]$$

$$\Rightarrow \quad \frac{d^2V}{dx^2} = \frac{\pi}{3}(-2R - 6x)$$

$$\Rightarrow \quad \left[\frac{d^2V}{dx^2}\right]_{x=\frac{R}{3}} = \frac{\pi}{3}\left[-2R - \frac{6R}{3}\right]$$

$$= \frac{\pi}{3}(-4R) = -\frac{4\pi R}{3} < 0$$

Thus, $\frac{d^2V}{dx^2} < 0 \Rightarrow V$ is maximum. **(1)**

dx^2

Now, volume of cone is $V = \frac{\pi}{3}(R^2 - x^2)(R + x)$

On putting $x = \frac{R}{3}$, we get

$$V = \frac{\pi}{3}\left[R^2 - \frac{R^2}{9}\right]\left[R + \frac{R}{3}\right]$$

$$= \frac{32\pi R^3}{81} = \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{8}{27}\text{ [Volume of sphere]}$$

$$\left[\because \text{ volume of sphere} = \frac{4}{3}\pi R^3\right]$$

Hence, volume of largest cone

$$= \frac{8}{27}\text{ [Volume of sphere]}$$ **(1)**

29. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, with its vertex at one end of the major axis.

HOTS; Delhi 2010C

Given equation of ellipse is

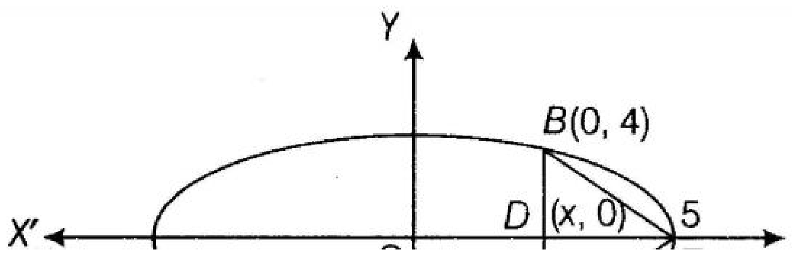
$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

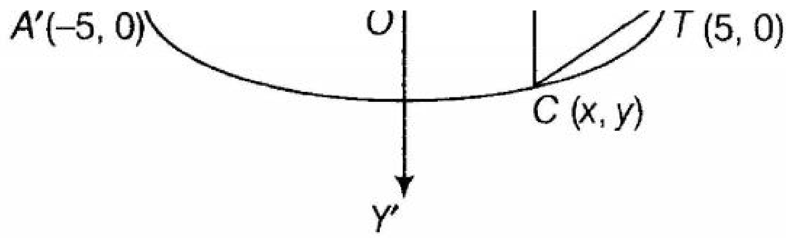
Here, $a = 5, b = 4$

$\therefore$ $a > b$

So, major axis is along X-axis.

Let ΔBTC be the isosceles triangle which is inscribed in the ellipse. And $OD = x$, $BC = 2y$ and $TD = 5 - x$.





Let A denotes the area of triangle. Then, we have

$$A = \frac{1}{2} \times \text{Base} \times \text{Height} = \frac{1}{2} \times BC \times TD$$

$$\Rightarrow \quad A = \frac{1}{2} \cdot 2y\,(5 - x)$$

$$\Rightarrow \quad A = y\,(5 - x) \quad \textbf{(1)}$$

On squaring both sides, we get

$$A^2 = y^2\,(5 - x)^2 \quad \ldots\text{(i)}$$

Now, $\frac{x^2}{25} + \frac{y^2}{16} = 1 \Rightarrow \frac{y^2}{16} = 1 - \frac{x^2}{25}$

$$\Rightarrow \quad y^2 = \frac{16}{25}(25 - x^2)$$

On putting value of y^2 in Eq. (i), we get

$$A^2 = \frac{16}{25}(25 - x^2)(5 - x)^2$$

Let $A^2 = Z$

Then, $$Z = \frac{16}{25}(25 - x^2)(5 - x)^2 \quad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dZ}{dx} = \frac{16}{25}[(25 - x^2)\,2\,(5 - x)(-1) + (5 - x)^2(-2x)]$$

$$= \frac{16}{25}(-2)(5 - x)^2(2x + 5)$$

$$= \frac{-32}{25}(5 - x)^2(2x + 5) \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dZ}{dx} = 0$

$$\Rightarrow -\frac{32}{25}(5 - x)^2(2x + 5) = 0 \Rightarrow x = 5 \text{ or } -\frac{5}{2} \quad \textbf{(1)}$$

Now, when $x = 5$, then

$$Z = \frac{16}{25}(25-25)(5-5)^2 = 0$$

which is not possible. So, $x = 5$ is rejected.

$$\therefore \quad x = -\frac{5}{2}$$

Now, $$\frac{d^2Z}{dx^2} = \frac{d}{dx}\left(\frac{dZ}{dx}\right) = \frac{d}{dx}\left[-\frac{32}{25}(5-x)^2(2x+5)\right]$$

$$= -\frac{32}{25}[(5-x)^2 \cdot 2 - (2x+5)\,2\,(5-x)]$$

$$= -\frac{64}{25}(5-x)(-3x) = \frac{192x}{25}(5-x)$$

At $x = \frac{-5}{2}$, $\left[\frac{d^2Z}{dx^2}\right]_{x=-\frac{5}{2}} < 0$

$\Rightarrow Z$ is maximum. **(1)**

$\therefore$ Area A is maximum, when $x = -\frac{5}{2}$ and $y = 12$.

Also, the maximum area

$$Z = A^2 = \frac{16}{25}\left(25 - \frac{25}{4}\right)\left[5 + \frac{5}{2}\right]^2$$

$$= \frac{16}{25} \times \frac{75}{4} \times \frac{225}{4} = 3 \times 225$$

Hence, the maximum area, $A = \sqrt{3 \times 225}$

$$= 15\sqrt{3} \text{ sq units}$$ **(1)**

NOTE If A is maximum/minimum, then A^2 is maximum/minimum.

30. Show that the right circular cylinder, open at the top and of given surface area and maximum volume is such that its height is equal to the radius of the base.

Delhi 2010; 2009C

Let V be the volume, S be the total surface area of a right circular cylinder which is open at the top. Again, let r be the radius of base and h be the height.

Now, $S = 2\pi rh + \pi r^2$

[$\because$ cylinder is open at top]

$$\Rightarrow \quad h = \frac{S - \pi r^2}{2\pi r} \quad \text{...(i)} \ \textbf{(1)}$$

Also, volume of cylinder is given by

$$V = \pi r^2 h$$

$$\Rightarrow \quad V = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r}\right) \quad \text{[using Eq. (i)]}$$

$$\Rightarrow \quad V = \frac{rS - \pi r^3}{2} \quad \textbf{(1)}$$

On differentiating w.r.t. r, we get

$$\frac{dV}{dr} = \frac{S - 3\pi r^2}{2} \quad \textbf{(1)}$$

For maxima and minima, put $\frac{dV}{dx} = 0$

$$\therefore \quad \frac{S - 3\pi r^2}{2} = 0 \Rightarrow S = 3\pi r^2$$

$$\Rightarrow \quad 2\pi rh + \pi r^2 = 3\pi r^2 \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad h = r \quad \textbf{(1)}$$

$\therefore$ Height of cylinder = Radius of the base

Also, $\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right) = \frac{d}{dr}\left(\frac{S - 3\pi r^2}{2}\right) = -\frac{6\pi r}{2}$

$$= -3\pi r < 0, \text{ as } r > 0 \quad \textbf{(1)}$$

Thus, $\frac{d^2V}{dr^2} < 0 \Rightarrow V$ is maximum.

Hence, volume of cylinder is maximum, when its height is equal to radius of the base. **(1)**

31. A manufacturer can sell x items at a price of ₹ $\left(5 - \frac{x}{100}\right)$ each. The cost price of x items is ₹ $\left(\frac{x}{5} + 500\right)$. Find the number of items he should sell to reach maximum profit.

HOTS; All India 2009

Given the manufacturer sells x items at price of ₹ $\left(5 - \frac{x}{100}\right)$ each.

∴ Total revenue obtained

$$= ₹\left[x\left(5 - \frac{x}{100}\right)\right] = ₹\left(5x - \frac{x^2}{100}\right) \quad \textbf{(1)}$$

Also, cost price of x items = ₹ $\left(\frac{x}{5} + 500\right)$

Let $P(x)$ be the profit function. Then, we know that

Profit = Revenue – Cost **(1)**

$$\therefore \quad P = \left(5x - \frac{x^2}{100}\right) - \left(\frac{x}{5} + 500\right)$$

$$\Rightarrow \quad P = \frac{-x^2}{100} + \frac{24x}{5} - 500 \quad \textbf{(1)}$$

On differentiating w.r.t. x, we get

$$\frac{dP}{dx} = \frac{-2x}{100} + \frac{24}{5}$$

For maxima and minima, put $\frac{dP}{dx} = 0$ **(1)**

$$\Rightarrow \quad \frac{-2x}{100} + \frac{24}{5} = 0$$

$$\Rightarrow \quad x = 240 \quad \textbf{(1)}$$

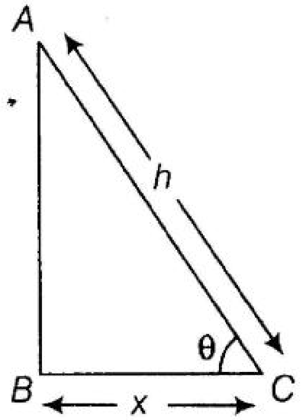
Also, $$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(\frac{dP}{dx}\right) = \frac{d}{dx}\left(-\frac{2x}{100} + \frac{24}{5}\right)$$
$$= -\frac{2}{100} = -\frac{1}{50} < 0$$

Thus, at $x = 240$, $\frac{d^2P}{dx^2} < 0 \Rightarrow P$ is maximum.

Hence, number of items sold to have maximum profit is 240. **(1)**

32. If the sum of the lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\pi/3$. HOTS; All India 2009

Let ABC be the right angled triangle with $BC = x$ and $AC = h$.



Now, given that $h + x = a$ ...(i)

where, a = constant **(1)**

And A denotes the area of triangle. Then,

$$A = \frac{1}{2} \times BC \times AB$$

$$\Rightarrow \quad A = \frac{1}{2} x \cdot \sqrt{h^2 - x^2}$$

$$\left[\begin{array}{l} \text{in right angled } \Delta ABC, \\ AB^2 = AC^2 - BC^2 = h^2 - x^2 \\ \therefore AB = \sqrt{h^2 - x^2} \end{array}\right] \quad \textbf{(1)}$$

On squaring both sides, we get

$$A^2 = \frac{x^2}{4}(h^2 - x^2) \Rightarrow A^2 = \frac{x^2}{4}[(a - x)^2 - x^2]$$

$$[\because h = a - x, \text{ from Eq. (i)}]$$

$$\Rightarrow \quad A^2 = \frac{a^2x^2 - 2ax^3}{4} \quad \textbf{(1)}$$

On differentiating both sides w.r.t. x, we get

$$2A \frac{dA}{dx} = \frac{1}{4}(2a^2x - 6ax^2) \quad \text{...(ii)}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{8A}(2a^2x - 6ax^2)$$

For maxima and minima, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{1}{8A}(2a^2x - 6ax^2) = 0$$

$$\Rightarrow \quad 2a^2x = 6ax^2 \Rightarrow x = \frac{a}{3} \qquad \textbf{(1/2)}$$

Again, differentiating Eq. (ii) w.r.t. x, we get

$$2A \cdot \frac{d^2A}{dx^2} + \frac{dA}{dx} \cdot 2\frac{dA}{dx} = \frac{1}{4}(2a^2 - 12ax)$$

$$\Rightarrow \quad 2A \cdot \frac{d^2A}{dx^2} + 2\left(\frac{dA}{dx}\right)^2 = \frac{1}{4}(2a^2 - 12ax)$$

On putting $\frac{dA}{dx} = 0$ and $x = \frac{a}{3}$, we get

$$2A\frac{d^2A}{dx^2} = \frac{1}{4}\left[2a^2 - 12a \times \frac{a}{3}\right]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{1}{8A}[2a^2 - 4a^2]$$

$$= -\frac{2a^2}{8A} = -\frac{a^2}{4A} < 0$$

$$\because \quad \frac{d^2A}{dx^2} < 0 \Rightarrow A \text{ is maximum.} \qquad \textbf{(1½)}$$

Also, in the given right angled ΔABC, we have

$$\cos\theta = \frac{BC}{AC} = \frac{x}{h} = \frac{x}{a - x} \qquad [\because h = a - x]$$

$$\therefore \quad \cos\theta = \frac{\left(\frac{a}{3}\right)}{\left(a - \frac{a}{3}\right)} = \frac{\left(\frac{a}{3}\right)}{\left(\frac{2a}{3}\right)} = \frac{a}{3} \times \frac{3}{2a} = \frac{1}{2}$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \cos\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$$

Hence, area of triangle is maximum, when $\theta = \frac{\pi}{3}$. **(1)**

33. A tank with rectangular base and rectangular sides, open at the top is to be constructed, so that its depth is 2 m and volume is 8 m^3. If building of tank cost ₹ 70 per sq m for the base and ₹ 45 per sq m for sides. What is the cost of least expensive tank?

HOTS; Delhi 2009

Let x m be the length, y m be the breadth and $h = 2$ m be the depth of the tank. Let ₹ H be the total cost for building the tank. Now, given that $h = 2$ m and volume of tank = 8 m^3

Also, area of the rectangular base of the tank

$= \text{Length} \times \text{Breadth} = xy \text{ m}^2$ **(1)**

and the area of the four rectangular sides

$= 2 \, (\text{Length} + \text{Breadth}) \times \text{Height}$

$= 2\,(x + y) \times 2 = 4\,(x + y) \text{ m}^2$ **(1)**

$\therefore$ Total cost, $H = 70 \times xy + 45 \times 4\,(x + y)$

$\Rightarrow \qquad H = 70xy + 180\,(x + y)$ ...(i)

Also, volume of tank = 8 m^3

$\Rightarrow \quad l \times b \times h = 8 \quad \Rightarrow x \times y \times 2 = 8$

$\Rightarrow \quad y = \dfrac{4}{x}$ ...(ii) **(1)**

On putting the value of y from Eq. (ii) in Eq. (i), we get

$$H = 70x \times \frac{4}{x} + 180\left(x + \frac{4}{x}\right)$$

$$\Rightarrow \quad H = 280 + 180\left(x + \frac{4}{x}\right) \quad \text{...(iii)}$$

On differentiating w.r.t. x, we get

$$\frac{dH}{dx} = 180\left(1 - \frac{4}{x^2}\right)$$

For maxima and minima, put $\dfrac{dH}{dx} = 0$

$$\Rightarrow 180\left(1 - \frac{4}{x^2}\right) = 0 \Rightarrow 1 - \frac{4}{x^2} = 0$$

$$\Rightarrow \quad \frac{4}{x^2} = 1 \Rightarrow x^2 = 4$$

$\Rightarrow \quad x = 2$ $[\because x > 0]$ **(1)**

Also, $\dfrac{d^2H}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dH}{dx}\right) = \dfrac{d}{dx}\left[180\left(1 - \dfrac{4}{x^2}\right)\right]$

$$= \frac{8}{x^3} \times 180$$

At $x = 2$, $\left[\dfrac{d^2H}{dx^2}\right]_{x=2} = \dfrac{8}{2^3} \times 180 = 180 > 0$

$\because \quad \dfrac{d^2H}{dx^2} > 0 \Rightarrow H$ is least at $x = 2$. **(1)**

Also, the least cost $= 280 + 180\left(2 + \dfrac{4}{2}\right)$

[put $x = 2$ in Eq. (iii) to get least cost H]

$= 280 + 180 \times 4 = 280 + 720 = ₹1000$

Hence, the cost of least expensive tank is ₹1000. **(1)**

34. Show that the height of the closed right circular cylinder, of given volume and minimum total surface area, is equal to its diameter. **All India 2008C**

> Here, we have two independent variables r and h, so we eliminate one variable. For this, find the value of h in terms of r and V and put in surface area, then use the second derivative test.

Let r be the radius of base, h be the height, V be the volume and S be the total surface area of the closed right circular cylinder. Then, given

$$V = \pi r^2 h$$

$$\Rightarrow \quad h = \frac{V}{\pi r^2} \quad \text{...(i)} \; \textbf{(1)}$$

Now, we know that, total surface area of cylinder is given by

$$S = 2\pi r^2 + 2\pi r h$$

$$\Rightarrow \quad S = 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right)$$

$$\left[\because h = \frac{V}{\pi r^2}, \text{ from Eq.(i)}\right]$$

$$\Rightarrow \quad S = 2\pi r^2 + \frac{2V}{r}$$

On differentiating w.r.t. r, we get

$$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2} \quad \textbf{(1½)}$$

For maxima and minima, put $\frac{dS}{dr} = 0$

$$\Rightarrow \quad 4\pi r - \frac{2V}{r^2} = 0 \quad \Rightarrow \quad V = 2\pi r^3$$

$$\Rightarrow \quad \pi r^2 h = 2\pi r^3 \quad [\because V = \pi r^2 h]$$

$$\Rightarrow \quad h = 2r$$

i.e. Height = Diameter of the base **(1½)**

Also, $\frac{d^2S}{dr^2} = \frac{d}{dr}\left(\frac{dS}{dr}\right) = \frac{d}{dr}\left(4\pi r - \frac{2V}{r^2}\right)$

$$= 4\pi + \frac{4V}{r^3} > 0, \text{ as } r > 0 \text{ and } V > 0$$

Thus, $\frac{d^2S}{dr^2} > 0$

$\Rightarrow S$ is minimum. **(1)**

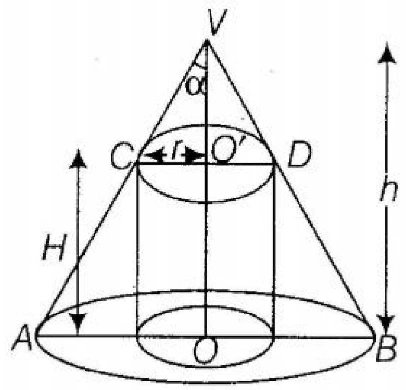
Hence, total surface area S is minimum, when height is equal to the diameter of the base. **(1)**

35. Show that the volume of the greatest cylinder can be inscribed in a cone of height h and semi-vertical angle α is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

All India 2008

Let VAB be the given cone of height h and semi-vertical angle α. Again, let V denotes the volume of the cylinder. From the figure, we have

(1/2)



Radius of base of cylinder $= O'C = r$

$H =$ Height of cylinder $= OO' = h - VO'$

Now, in right angled $\Delta VO'C$, we have

$$\tan \alpha = \frac{O'C}{VO'} = \frac{r}{VO'} \quad [\because O'C = r]$$

$$\Rightarrow \quad VO' = \frac{r}{\tan \alpha}$$

$$\Rightarrow \quad VO' = r \cot \alpha$$ **(1)**

$\therefore$ Height of cylinder, $H = OO'$

$$= h - VO' = h - r \cot \alpha$$

Now, volume of cylinder is given by

$$V = \pi r^2 H$$

$$\Rightarrow \quad V = \pi r^2 (h - r \cot \alpha) \quad \ldots\text{(i)}$$

$[\because H = h - r \cot \alpha]$

$$\Rightarrow \quad V = \pi r^2 h - \pi r^3 \cot \alpha$$

On differentiating w.r.t. r, we get

$$\frac{dV}{dr} = 2\pi rh - 3\pi r^2 \cot \alpha$$ **(1½)**

For maxima and minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow 2\pi rh - 3\pi r^2 \cot\alpha = 0$$

$$\Rightarrow \quad r = \frac{2h}{3}\tan\alpha \quad \textbf{(1)}$$

Also, $$\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right)$$

$$= \frac{d}{dr}(2\pi rh - 3\pi r^2 \cot\alpha)$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} = 2\pi h - 6\pi r \cot\alpha$$

At $r = \frac{2h}{3}$, $\left[\frac{d^2V}{dr^2}\right]_{r=\frac{2h}{3}\tan\alpha}$

$$= 2\pi h - 6\pi \cot\alpha\left(\frac{2h}{3}\tan\alpha\right)$$

$$= 2\pi h - 4\pi h \tan\alpha \cot\alpha$$

$$= 2\pi h - 4\pi h$$

$[\because \tan\alpha \cot\alpha = 1]$

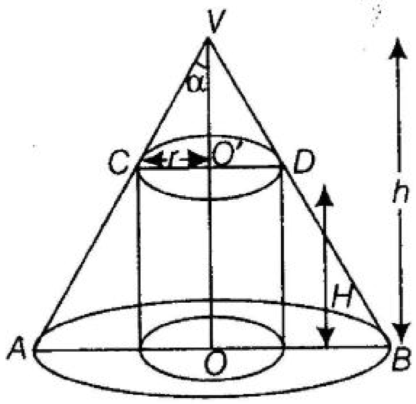
$$= -2\pi h < 0 \text{ as } h > 0 \quad \textbf{(1)}$$

Thus, $\frac{d^2V}{dr^2} < 0 \Rightarrow V$ is maximum.

36. Show that the height of the cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}h$.

Delhi 2008

Let VAB be the given cone of height h and a semi-vertical angle α. Again let V denotes the volume of cylinder. From the figure, we have



(1/2)

H = Height of cylinder $= OO' = h - VO'$

Now, in right angled $\Delta VO'C$, we get

$$\tan\alpha = \frac{O'C}{VO'} = \frac{r}{VO'}$$

$$\Rightarrow \qquad VO' = \frac{r}{\tan\alpha} = r\cot\alpha \qquad \textbf{(1)}$$

$\therefore$ Height of cylinder $= H$

$$= h - VO' = h - r\cot\alpha$$

Also, radius of base of cylinder $= O'C = r$

∴ Volume of cylinder is given by

$$V = \pi r^2 H \Rightarrow V = \pi r^2 (h - r\cot\alpha)$$

$$[\because H = h - r\cot\alpha]$$

$$\Rightarrow \quad V = \pi r^2 h - \pi r^3 \cot\alpha$$

On differentiating w.r.t. r, we get

$$\frac{dV}{dr} = 2\pi rh - 3\pi r^2 \cot\alpha$$

For maxima and minima, put $\frac{dV}{dr} = 0$

$$\Rightarrow \quad 2\pi rh - 3\pi r^2 \cot\alpha = 0$$

$$\Rightarrow \quad r = \frac{2h}{3}\tan\alpha \qquad \textbf{(1½)}$$

Now,
$$\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{dV}{dr}\right)$$
$$= \frac{d}{dr}(2\pi rh - 3\pi r^2\cot\alpha)$$
$$= 2\pi h - 6\pi r\cot\alpha$$

At
$$r = \frac{2h}{3}\tan\alpha, \left[\frac{d^2V}{dr^2}\right]_{r=\frac{2h}{3}\tan\alpha}$$
$$= 2\pi h - 6\pi\cot\alpha \cdot \frac{2h}{3}\tan\alpha$$
$$= 2\pi h - 4\pi h\tan\alpha\cot\alpha$$
$$= 2\pi h - 4\pi h \quad [\because \tan\alpha\cot\alpha = 1]$$
$$= -2\pi h < 0 \text{ as } h > 0$$

Thus, $\frac{d^2V}{dr^2} < 0 \Rightarrow$ Volume is maximum. **(1½)**

Now, height of cylinder, $H = h - r\cot\alpha$

$$= h - \frac{2h}{3}\tan\alpha\cot\alpha \qquad \left[\because r = \frac{2h}{3}\tan\alpha\right]$$

$$= h - \frac{2h}{3} = \frac{h}{3} \qquad [\because \tan\alpha\cdot\cot\alpha = 1]$$

Hence, height of cylinder of maximum volume that can be inscribed in a cone of height h is $\frac{1}{3}h$. **(1½)**